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Contributions To Some Classes Of Difference-Ratio-Type Exponential Estimators For Population Mean Under Simple Random Sampling

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***Abstract:** The estimators based on dual use of auxiliary variables are recently developed as they provide more better population mean estimates than the use of auxiliary variably only. In this study, we propose some classes of estimators based on dual use of auxiliary information for population mean under simple random sampling. The idea is to transform the exponent of the auxiliary variable using suitable bounded constants. The mathematical expression for the bias and mean squared errors (MSE) of the proposed estimators are obtained. The performance evaluation of the proposed estimator is comprehensively explored and compared with the existing different-ratio-type exponential estimators in terms of MSE and percentage relative efficiency (PRE) for various choice of $\alpha[0.1, 1]$. It is found that the proposed estimators are able to perform substantially better than existing estimators. Both real and simulated population datasets are considered to support the theory.*

***Keywords:** Bias, Mean-squared error, Auxiliary variable, Ratio-exponential estimator, Percentage relative efficiency.*

I. INTRODUCTION

In the last few decades, survey sampling has been widely used in different fields of operations including agriculture, business management, demography, economics, education, engineering, industry, medical sciences, political sciences, social sciences, and among others. As a result, many classes of estimators for population mean under simple random sampling that depend on auxiliary information has been developed to provide an efficient estimator. In many such techniques and applications, the classical ratio, product and regression estimators are broadly used for estimating the unknown population parameters, provided that suitable correlation is existing between the variable of interest and the auxiliary variable. In the past, many authors have proposed different ratio-type estimators by appropriately modifying the auxiliary variables (see Gupta and Shabbir (2008), Kadilar and Cingi (2004), Kadilar and Cingi (2006a), Kadilar and Cingi (2006b), Haq and Shabbir (2013), Singh and Solanki (2013), Grover and Kaur (2014), Shabbir et al. (2014), and references cited therein for research findings). As a matter of fact, the proper use of auxiliary information in sampling process leads to reduction in variance of estimator for unknown population parameter(s).

The existing methods of estimation are based only on usual form of supplementary information of auxiliary variable(s). However, the idea of incorporating additional information have recently introduced in literature (Haq et al., 2017; Irfan et al., 2020) to enhance the efficiency of the estimators. This additional information is provided from the rank of the auxiliary variable and refer to as rank of auxiliary variable. This study is intended to discover more efficient

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estimators using both auxiliary information and the rank of auxiliary information. In this study, we transform the exponent of the auxiliary variable of generalized ratio type estimator developed by Singh et al. (2009) to propose different-ratio-type estimators in direction of Haq et al. (2017) and Irfan et al. (2020) for estimation of population mean under simple random sampling schemes.

The rationale of the newly proposed estimator is to modify the auxiliary variable x as: $x' = \mathfrak{a}x + \frac{1}{2}[(\alpha - 1)\mathfrak{a}X + (\alpha + 1)\mathfrak{b}]$ which was defined earlier by Singh et al. (2009) as $x' = \mathfrak{a}x + \mathfrak{b}$. The essence of this is to systematically smoothing the auxiliary variable to enhance the efficiency of the existing estimators. In this case, α is the constant values bounded between 0.1 and $1 (i.e., 0 < \alpha \leq 1)$.

Suppose a random sample of size n is drawn from a finite population $\Omega = (\Omega_1, \Omega_2, \cdots, \Omega_N)$ of size N by a simple random sampling without replacement (SRSWOR) method. Let $y_{i,}\, x_i$ and r_{xi} be the value of the study variable, Y, auxiliary variable, X and the rank of auxiliary variable, R_x. Some useful parameters are presented in Table 1.1.

Parameters	Study variable	Auxiliary variable	Rank of auxiliary variable
Sample mean	$\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$	$\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$	$\bar{r}_x = n^{-1}\sum_{i=1}^{n} r_{xi}$
Population mean	$\bar{Y} = N^{-1}\sum_{i=1}^{n} y_i$	$\bar{X} = N^{-1}\sum_{i=1}^{n} x_i$	$\bar{R}_x = N^{-1}\sum_{i=1}^{n} r_{xi}$
Population variance	$S_Y^2 = (N-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{Y})^2$	$S_X^2 = (N-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{X})^2$	$S_{R_x}^2 = (N-1)^{-1}\sum_{i=1}^{n} r_{xi} - \bar{R}_x)^2$
Sample variance	$S_Y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$	$S_X^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$	$S_{R_x}^2 = (n-1)^{-1}\sum_{i=1}^{n} r_{xi} - \bar{r}_x)^2$
Coefficient of variation	$C_y = S_y/\bar{Y}$	$C_x = S_y/\bar{X}$	$C_{r_x} = S_{r_x}/\bar{R}_x$

Table 1.1: Some Useful Parameters

The covariance of y and x, y and r_x, x and r_x are $S_{yx} = \frac{\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})}{N-1}$, $S_{yr_x} = \frac{\sum_{i=1}^{N}(y_i - \bar{Y})(r_{xi} - \bar{R}_x)}{N-1}$, and $S_{xr_x} = \frac{\sum_{i=1}^{N}(x_i - \bar{X})(r_{xi} - \bar{R}_x)}{N-1}$, respectively.

The rest of the study is arranging as follows. In Section 2, we consider the brief review of some classical and recent estimators of the finite population mean. In Section 3, an enhanced estimator is proposed for effectively estimating the finite population mean. In order to study and investigate the performances of the considered estimators, a numerical study is carried out in Section 4 using simulated population and different finite real population datasets. Finally, we give the conclusion of the study in Section 5.

II. EXISTING ESTIMATORS IN LITERATURE

In this section the brief introduction of existing estimators under simple random sampling without replacement (SRSWOR): unbiased mean per unit, ratio, product, regression and the well-known exponential type estimators are given alongside with their respective variance and MSE.

Unbiased mean per unit: $\hat{\bar{Y}} = \bar{y}$ with variance

$$var(\bar{y}) = \lambda \bar{Y}^2 C_y^2 \qquad (2.1)$$

The usual ratio and product estimators were developed by Corchran (1940) and Muthy (1964), respectively, are given by

$$\hat{\bar{Y}}_R = \bar{y}\frac{\bar{X}}{\bar{x}}, \quad \hat{\bar{Y}}_P = \bar{y}\frac{\bar{x}}{\bar{X}} \qquad (2.2)$$

It is obvious that, irrespective of the unbiasedness, the usual ratio or product estimator is more efficient than unbiased mean per unit estimator. Their MSEs are given by

$$MSE\left(\hat{\bar{Y}}_R^2\right) = \lambda \bar{Y}^2 (C_y^2 + C_x^2 - 2\rho_{yx} C_y C_x) \qquad (2.3)$$

And $MSE\left(\hat{\bar{Y}}_P^2\right) = \lambda \bar{Y}^2 (C_y^2 + C_x^2 + 2\rho_{yx} C_y C_x) \qquad (2.4)$

A number of transformed ratio-estimators for estimating the finite population mean have been suggested by several authors using auxiliary information. Such as Sisodia and Dwivedi (1981), Bedi (1996), Upadhyaya and Singh (1999), Singh (2003a), Singh and Tailor (2003) and many others. Khoshnevisan et al. (2007) proposed a general class of estimators that comprises some modified ratio-type estimators as special cases, given by

$$\hat{\bar{Y}}_k = \bar{y}\left\{\frac{a\bar{X} + b}{\alpha(a\bar{x} + b) + (1-\alpha)(a\bar{X} + b)}\right\}^g, \qquad (2.5)$$

where $a (\neq 0)$ and b are either known constants or functions of any known population parameters, such as coefficient of variation C_x coefficient of skewness $\beta_2(x)$, correlation coefficient ρ_{yx}, etc. Note that when $g = 0, 1$ and -1, the mean per unit, ratio and product estimators are the special cases of (6) respectively, given that α = = a 1 and b = 0. The minimum MSE of (5) at the optimum value of $\alpha\theta g$, where $\theta = \frac{a\bar{X}}{a\bar{X} + b}$, is given by

$$MSE_{min}\left(\hat{\bar{Y}}_k\right) \cong \lambda \bar{Y}^2 C_y^2 (1 - \rho_{yx}^2),$$

(1) Classical different estimator is

$$\hat{\bar{Y}}_D = \bar{y} + k(\bar{x} - \bar{X}) \qquad (2.7)$$

where k is an unknown constant. It is easy to show that (2.7) is unbiased. The minimum variance of (2.7) at the optimum value of k, i.e., $k_{opt} = \rho_{yx}\frac{S_y}{S_x}$ is given by

$$Var_{min}\left(\hat{\bar{Y}}_D\right) \cong \lambda \bar{Y}^2 C_y^2 (1 - \rho_{yx}^2) \qquad (2.8)$$

This is equivalent to asymptotic variance of regression $\hat{\bar{Y}}_{lr} = \bar{y} + b(\bar{x} - \bar{X})$, where b is the slope estimator of the population regression coefficient $\beta = k_{opt}$. The difference estimator (2.7) is always better (in terms of relative efficiency) than the usual ratio and product estimators (2.2) when estimating $\bar{Y}$.

Rao (1991) proposed an improved different-type estimator given by

$$\hat{\bar{Y}}_{R.D} = t_1\bar{y} + t_2(\bar{x} - \bar{X}) \qquad (2.9)$$

where t_1 and t_2 are suitable selected constant. The minimum MSE of (9) at optimum $t_{1(opt)} = [1 + \lambda C_x^2(1 - \rho_{yx}^2)]^{-1}$ and $t_{2(opt)} = \bar{Y}\lambda C_y \rho_{yx} \bar{X} C_x\, [\{1 + \lambda C_x^2(1 - \rho_{yx}^2)\}]^{-1}$ given by

$$\hat{\bar{Y}}_{R.D} \cong \frac{\lambda \bar{Y}^2 C_y^2 (1 - \rho_{yx}^2)}{1 + \lambda C_x^2 (1 - \rho_{yx}^2)} \qquad (2.10)$$

It is easy to show that estimator $\hat{\bar{Y}}_{R.D}$ is highly efficient than estimator $\hat{\bar{Y}}_D$

Bahl and Tuteja (1991) suggested ratio- and product-type exponential estimators, given by

$$\hat{\bar{Y}}_{BT.R} = \bar{y}exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right] \qquad (2.11)$$

$$\hat{\bar{Y}}_{BT.P} = \bar{y}exp\left[\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right] \qquad (2.12)$$

The MSE of (2.11) and (2.12), respectively, are given by

$$MSE\left(\hat{\bar{Y}}_{BT.R}\right) = \lambda\left(C_y^2 + \frac{1}{4}C_x^2 - \rho_{yx} C_y C_x\right) \qquad (2.13)$$

$$MSE\left(\hat{\bar{Y}}_{BT.P}\right) = \lambda\left(C_y^2 + \frac{1}{4}C_x^2 + \rho_{yx} C_y C_x\right) \qquad (2.14)$$

The average of (2.11) and (2.12) are given by

$$\hat{\bar{Y}}_{BT.avg} = \frac{\bar{y}}{2}\left[exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] + exp\left[\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right]\right]$$

In the line of Bahl and Tuteja (1991), Singh et al. (2009) proposed a generalized ratio-type exponential estimator,

$$\hat{\bar{Y}}_S = \bar{y}exp\left[\frac{a(\bar{X}-\bar{x})}{a(\bar{X}+\bar{x})+2b}\right] \quad (2.15)$$

It is clear that both estimators $\hat{\bar{Y}}_{BT.P}$ and $\hat{\bar{Y}}_{BT.R}$ are special cases of $\hat{\bar{Y}}_S$ when $(a = 1, b = 0)$ and $and\ (a = -1,\ b = 0)$ respectively. Likewise, when $a = 1,\ b = N\bar{X}$, we have Bedi (1996)'s transformed ratio-type exponential estimator given by

$$\hat{\bar{Y}}_{BT.B} = \bar{y}exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2N\bar{X}}\right] \quad (2.16)$$

The minimum MSE of $\hat{\bar{Y}}_S$ turns out to be equivalent to min variance of $\hat{\bar{Y}}_D$, i.e., $MSE_{min}\left(\hat{\bar{Y}}_S\right) \cong \lambda\bar{Y}^2C_y^2(1-\rho_{yx}^2)$

In the direction of Bahl and Tuteja (1991), Singh et al. (2009), and Rao (1991), Shabbir and Gupta (2010) proposed a -type exponential estimator by integrating $\hat{\bar{Y}}_{R.D}$ and $\hat{\bar{Y}}_{BT.B}$ given by

$$\hat{\bar{Y}}_{SG} = u_1\bar{y} + u_2(\bar{x}-\bar{X})exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2N\bar{X}}\right] \quad (2.17)$$

where u_1 and u_2 are choice of suitable constants.

Following the work of Shabbir and Gupta (2010), a similar estimator was proposed by Grover and Kaur (2011) by integrating $\hat{\bar{Y}}_{R.D}$ and $\hat{\bar{Y}}_{BT.R}$ given by

$$\hat{\bar{Y}}_{GK} = v_1\bar{y} + v_2(\bar{x}-\bar{X})exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \quad (2.18)$$

where v_1 and v_2 are choice of suitable constants.

Grover and Kaur (2014) proposed a generalized class of ratio-type exponential estimators by integrating $\hat{\bar{Y}}_{R.D}$ and $\hat{\bar{Y}}_S$, given by

$$\hat{\bar{Y}}_{GK.G} = w_1\bar{y} + w_2(\bar{x}-\bar{X})exp\left[\frac{a(\bar{X}-\bar{x})}{a(\bar{X}+\bar{x})+2b}\right] \quad (2.19)$$

where w_1 and w_2 are choice of suitable constants. Note that estimators given by (2.17) and (2.18) hold within (2.19). Thus, at the optimum values, $w_{1(opt)} = \frac{1-\frac{1}{2}\lambda\theta^2C_x^2}{1+\lambda C_y^2(1-\rho_{yx}^2)}$ and $w_{2(opt)} = \frac{\bar{Y}\left[\lambda\theta^3C_x^3+C_y\rho_{yx}-\frac{1}{2}\lambda\theta^2C_x^2C_y\rho_{yx}-\theta C_x\left(1-\lambda C_y^2(1-\rho_{yx}^2)\right)\right]}{\bar{X}C_x[1+\lambda C_y^2(1-\rho_{yx}^2)]}$ the minimum MSE of estimator in (2.19) given by

$$MSE\left(\hat{\bar{Y}}_{GK.G}\right) = \frac{\lambda\bar{Y}^2\left[C_y^2(1-\rho_{yx}^2)-\frac{1}{4}\lambda\theta^4C_x^2-\lambda\theta^2C_x^2C_y^2(1-\rho_{yx}^2)\right]}{1+C_y^2(1-\rho_{yx}^2)} \quad (2.20)$$

In the direction of the estimators $\hat{\bar{Y}}_{SG}$, $\hat{\bar{Y}}_{GK}$ and $\hat{\bar{Y}}_{GK.G}$g, Haq et al. (2017) proposed a -type exponential estimator. this estimator based on the dual use of auxiliary variable and given by

$$\hat{\bar{Y}}_H = [\omega_1\bar{y} + \omega_2(\bar{X}-x) + \omega_3(\bar{R}_x-\bar{r}_x)]\ exp\left(\frac{a(\bar{X}-\bar{x})}{a(\bar{X}+\bar{x})+b}\right)$$

(2. 21)

where, $\bar{y}, \bar{x}$ and $\bar{r}_x$ are sample mean of study variable, auxiliary variable, and rank of auxiliary variable respectively, $\bar{X}$ and $\bar{R}_x$ are population mean of auxiliary variables, $a(\neq 0)$, and b are either suitable constant or function of auxiliary variable parameters and ω_1, ω_2 and $\omega_3$are the optimum value which minimizes the MSE of the estimator which comprises unknown parameters. The minimum mean squared error (MSE) of estimator $\hat{\bar{Y}}_H$ at $\omega_{1(opt)} = \frac{1-\frac{1}{2}\lambda\theta^2C_x^2}{1+\lambda C_y^2(1-R_{y.xr_x}^2)}$,

$$\omega_{2(opt)} = \frac{\bar{Y}\begin{bmatrix}\lambda\theta^3C_x^2(-1+\rho_{xr_x}^2)-C_y\left(1-\frac{1}{2}\lambda\theta^2C_x^2\right)(\rho_{yx}-\rho_{xr_x}\rho_{yr_x})\\ +\theta C_x(-1+\rho_{xr_x}^2)[-1+\lambda C_y^2(1-R_{y.xr_x}^2)]\end{bmatrix}}{\bar{X}C_x(-1+\rho_{xr_x}^2)[1+\lambda C_y^2(1-R_{y.xr_x}^2)]}$$ and

$$\omega_{3(opt)} = \frac{\bar{Y}\left(1-\frac{1}{2}\lambda\theta^2C_x^2\right)C_y(\rho_{xr_x}\rho_{yx}-\rho_{yr_x})}{\bar{R}C_{r_x}(-1+\rho_{xr_x}^2)[1+\lambda C_y^2(1-R_{y.xr_x}^2)]}$$ given by

$$MSE_{min}\left(\hat{\bar{Y}}_H\right) \cong \frac{\lambda\bar{Y}^2\left\{C_y^2(1-R_{y.xr_x}^2)-\frac{1}{4}\lambda\theta^4C_x^4-\lambda\theta^2C_x^2C_y^2(1-R_{y.xr_x}^2)\right\}}{1+C_y^2(1-R_{y.xr_x}^2)} \quad (2.22)$$

where $R_{y.xr_x}^2 = \frac{\rho_{yx}^2+\rho_{yr_x}^2-2\rho_{yx}\rho_{yr_x}\rho_{xr_x}}{1-\rho_{xr_x}^2}$ is the coefficient of multiple determination of Y on X and R_x in simple random sampling and $\theta = \frac{a\bar{X}}{2(a\bar{X}+b)}$.

In addition, Irfan (2020) proposed an efficient estimator following the lines of Haq et al. (2017). This estimator is based on both auxiliary variable and its rank.

$$\hat{\bar{Y}}_I = \frac{\pi_1}{2}\left(\frac{\bar{X}}{\bar{x}}+\frac{\bar{x}}{\bar{X}}\right)\hat{\bar{Y}}_{BT,avg} + \pi_2(\bar{R}_x-\bar{r}_x) + \pi_3(\bar{X}-\bar{x})\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \quad (2.23)$$

where $\hat{\bar{Y}}_{BT,avg}$ is the average of ratio and product exponential estimators, $\pi_1, \pi_2\ and\ \pi_3$ are suitable chosen constants. The minimum MSE of estimator $\hat{\bar{Y}}_I$ at $\pi_1 = \frac{F_1F_2-2\lambda^2C_x^2C_r^2F_3}{F_2F_4-2F_3^2}$,

$\pi_{2(opt)} = \frac{2\lambda\rho_{yr}C_yC_r(F_1F_2-2\lambda^2C_x^2C_r^2F_3)+\lambda\rho_{xr}C_xC_r(F_1F_3-\lambda^2C_x^2C_r^2F_4)}{\lambda C_r^2\frac{\bar{R}x}{\bar{Y}}(F_2F_4-F_3^2)}$ and $\pi_{3(opt)} = \frac{\lambda^2C_x^2C_r^2F_4-F_1F_3}{2\frac{\bar{X}}{\bar{Y}}(F_2F_4-2F_3^2)}$

Where $F_1 = \lambda C_r^2(8+5\lambda C_x^2;\quad F_2 = \lambda^2C_r^2C_x^2(1-\rho_{xr}^2);$ $F_3 = \lambda^2C_r^2[2C_yC_x(\rho_{yr}\rho_{xr}-\rho_{yx})+C_x^2];$ $F_4 = \lambda C_r^2[8+8\lambda C_y^2(1-\rho_{yr}^2)+10\lambda C_x^2]$

$$\begin{aligned}MSE_{min}\left(\hat{\bar{Y}}_I\right) = \frac{\bar{Y}^2}{\lambda C_r^2G_1^2}&[\lambda C_r^2G_1^2\\ &+\left(4\lambda C_r^2\left(1+\lambda C_y^2(1-\rho_{yr}^2)\right)\right.\\ &\left.+5\lambda^2C_x^2C_r^2\right)G_2^2\\ &\lambda^2C_x^2C_r^2(-\rho_{xr}^2)G_3^2 - C_r^2(8+5C_r^2)G_1G_3\\ &+(4\lambda C_r^2C_yC_x\\ &(\rho_{yr}\rho_{xr}-\rho_{yx})+\lambda C_x^2)G_2G_3\\ &-2\lambda^2C_x^2C_r^2G_1G_3]\end{aligned} \quad (2.24)$$

where $G_1 = F_2E_4 - 2F_3^2,\quad G_2 = F_1F_2 - 2\lambda^2C_x^2C_r^2F_3$ and $G_3 = \lambda^2C_x^2C_r^2F_4 - F_1F_3$

III. THE PROPOSED ESTIMATOR

Suppose α is a suitable choice of value bounded between 0 and 1, i.e., $0 < \alpha \leq 1$. $\mathfrak{a}(\neq 0)$ and $\mathfrak{b}$ is either any known constants or functions of any known population parameters of auxiliary variable x. These include standard deviation S_x, , coefficient of variation, C_x, coefficient of skewness, $\beta_1(x)$, coefficient of kurtosis, $\beta_2(x)$, coefficient of correlation, ρ_{yx} etc. we defined the original auxiliary variable x by

$$x' = \mathfrak{a}x + \frac{1}{2}[(\alpha-1)\mathfrak{a}X + (\alpha+1)\mathfrak{b}] \quad (3.1)$$

Thus, we have

$$\bar{X}' = \mathfrak{a}\bar{X} + \frac{1}{2}[(\alpha-1)\mathfrak{a}\bar{X} + (\alpha+1)\mathfrak{b}] \text{ and } \bar{x}' = \mathfrak{a}\bar{x} + \frac{1}{2}[(\alpha-1)\mathfrak{a}\bar{X} + (\alpha+1)\mathfrak{b}] \quad (3.2)$$

where $\bar{X}'$ and $\bar{x}'$ are population mean and sample mean of the new transformed auxiliary variable $\bar{x}'$ respectively. Here, the value of $\bar{X}'$ is known to us because $\mathfrak{a}$, $\mathfrak{b}$, and $\bar{X}$ are assumed to be known ahead.

In order to derive the bias, mean squared error (MSE), and minimum MSE of the proposed estimators we define the relative error and their expectation as follows; $\bar{y} = \bar{Y}(1+e_0)$, $\bar{r}_x = \bar{R}_x(1+e_1)$, $\bar{x} = \bar{X}(1+e_2)$, such that; $E(e_0) = E(e_1) = E(e_2) = 0$, $E(e_0^2), = \lambda C_y^2$, $E(e_1^2), = \lambda C_x^2$, $E(e_2^2), = \lambda C_{r_x}^2$, $E(e_0e_1), = \lambda\rho_{yx}C_yC_x$, $E(e_0e_2), = \lambda\rho_{yr_x}C_yC_{r_x}$, $E(e_1e_2), = \lambda\rho_{xr_x}C_xC_{R_x}$, where $\lambda = \left(\frac{1}{n} - \frac{1}{N}\right)$, $C_y^2 = \frac{S_y^2}{\bar{Y}^2}$, $C_x^2 = \frac{S_x^2}{\bar{X}^2}$, $C_{r_x}^2 = \frac{S_{r_x}^2}{\bar{R_x}^2}$, $\rho_{yx} = \frac{S_{yx}}{S_yS_x}$, $\rho_{yr_x} = \frac{S_{yr_x}}{S_yS_{r_x}}$, $\rho_{xr_x} = \frac{S_{xr_x}}{S_xS_{r_x}}$.

In the line of Singh et al. (2009), a ratio-type exponential estimator of finite population means that uses auxiliary information on X given by

$$\hat{\bar{Y}}_{(\alpha)} = \bar{y}exp\left(\frac{\bar{X}'-\bar{x}'}{\bar{X}'+\bar{x}'}\right) \quad (3.3)$$

Substitute (3.2) into (3.3), the estimator become

$$\hat{\bar{Y}}_{(\alpha)} = \bar{y}exp\left(\frac{\mathfrak{a}(\bar{X}-\bar{x})}{\mathfrak{a}(\bar{X}+\bar{x})+(\alpha-1)\mathfrak{a}\bar{X}+(\alpha+1)\mathfrak{b}}\right) \quad (3.4)$$

where α is a suitable choice of value bounded between 0 and 1, i.e., $0 < \alpha \le 1$. $\mathfrak{a}(\neq 0)$ and $\mathfrak{b}$ is either any known constants. It is obvious that ratio-type exponential estimators given in (11), (12), and (15) are special cases of $\hat{\bar{Y}}_{(\alpha)}$ when ($\alpha = 1$, $\mathfrak{a} = 1$, b=0), ($\alpha = 1$, $\mathfrak{a} = -1$, b=0) and ($\alpha = 1$), respectively.

3.1 First Proposed Estimator

Following Haq et al. (2017), we propose the first estimator which is different ratio-type exponential estimator given by

$$\hat{\bar{Y}}_{\alpha,1} = [\gamma_1\bar{y} + \gamma_2(\bar{X} - x) + \gamma_3(\bar{R}_x - \bar{r}_x)]\, exp\left(\frac{\mathfrak{a}(\bar{X}-\bar{x})}{\mathfrak{a}(\bar{X}+\bar{x})+(\alpha-1)\mathfrak{a}\bar{X}+(\alpha+1)\mathfrak{b}}\right) \quad (3.5)$$

where γ_1, γ_2 *and* γ_3 are aptly chosen constant to be determined.

Now we rewrite (3.5) in term of relative errors that was earlier defined, we have

$$\hat{\bar{Y}}_{\alpha,1} = [\gamma_1\bar{Y}(1+e_0) - \gamma_2\bar{X}e_1 - \gamma_3\bar{R}_xe_2]\left[1 - \theta e_1 + \frac{3}{2}\theta^2e_1^2 + \cdots\right] \quad (3.6)$$

where $\theta = \frac{\alpha\bar{X}}{(\alpha+1)[\mathfrak{a}\bar{X}+\mathfrak{b}]}$ is a known quantity since $\mathfrak{a}$, $\mathfrak{b}$, and $\bar{X}$ are previously assumed to be known quantities ahead. We expand (3.6) and keep the term up to second degree in errors and subtract $\bar{Y}$ from both sides, we have

$$\hat{\bar{Y}}_{\alpha,1} - \bar{Y} \cong -\bar{Y} + \bar{Y}\gamma_1 + \bar{Y}\gamma_1e_0 - \bar{Y}\gamma_1\theta e_1 - \bar{Y}\gamma_1\theta e_0e_1 + \frac{3}{2}\bar{Y}\gamma_1\theta^2e_1^2 - \gamma_2\bar{X}e_1 - \gamma_3\bar{R}_xe_2 + \gamma_2\bar{X}e_1^2 + \gamma_3\bar{R}_xe_1e_2 \quad (3.7)$$

Taking the expectation of (3.7) under first order of approximation, the bias of estimator $\hat{\bar{Y}}_{\alpha,1}$ is given as

$$Bias\left(\hat{\bar{Y}}_{\alpha xr}\right) \cong E\left(\hat{\bar{Y}}_{\alpha xr} - \bar{Y}\right)$$
$$\cong \bar{Y}\left[(\gamma_1 - 1) + \lambda\gamma_1\theta C_x\left(\frac{3}{2}\theta C_x - \rho_{yx}C_y\right) + \lambda\theta C_x\left(\gamma_2R_1C_x + \gamma_3R_2C_{r_x}\rho_{xr_x}\right)\right] \quad (3.8)$$

where $R_1 = \bar{X}/\bar{Y}$, $R_1 = \bar{R}_x/\bar{Y}$

Obtain the square of (3.7) under first order of approximation and take its expectation, the mean squared error (MSE) given as follows

$$\left(\hat{\bar{Y}}_{\alpha,1} - \bar{Y}\right)^2 = \bar{Y}^2 + \bar{Y}^2\gamma_1^2 + \bar{Y}^2\gamma_1^2e_0^2 + 4\bar{Y}^2\gamma_1^2\theta^2e_1^2 + \bar{X}^2\gamma_2^2e_1^2 + R_x^2\gamma_3^2e_2^2 - 4\bar{Y}^2\gamma_1\theta e_1e_2 + 4\bar{Y}\bar{X}\gamma_1\gamma_2\theta e_1^2 + 4\bar{Y}\bar{R}_x\gamma_1\gamma_3\theta e_1e_2 - 2\bar{Y}\bar{X}\gamma_1\gamma_2e_0e_1 - 2\bar{Y}R_x\gamma_1\gamma_3e_0e_2 \quad (3.9)$$
$$+2\bar{Y}\bar{X}\bar{R}_x\gamma_2\gamma_3e_1e_2 - 2\bar{Y}^2\gamma_1 - 3\bar{Y}^2\gamma_1\theta^2e_1^2 - 2Y^2\gamma_1\theta e_0e_1 - 2\bar{Y}\bar{X}\gamma_2\theta e_1^2 - 2\bar{Y}\bar{X}\gamma_3\theta e_1e_2$$

$$MSE(\hat{\bar{Y}}_{\alpha,1}) = \bar{Y}^2\left[1 + \gamma_1^2\{1 + \lambda(C_y^2 + 4\theta(\theta C_x^2 - \rho_{yx}C_y)\} - \lambda R_1\gamma_2C_x^2(2\theta - R_1\gamma_2)\right. \quad (3.10)$$
$$+\lambda R_2^2\gamma_3^2C_r^2 - 2\lambda R_1\gamma_1\gamma_2(\rho_{yx}C_yC_x - 2\theta C_x^2) - 2\lambda R_2\gamma_1\gamma_3C_r(\rho_{yr}C_y + 2\rho_{xr}\theta C_x)$$
$$+2\lambda R_1R_2\gamma_2\gamma_3\rho_{xr}C_xC_r - 2\lambda R_1\gamma_3\theta\rho_{xr}C_xC_r - 2\gamma_1\left\{1 + \lambda\theta C_x\left(\frac{3}{2}\theta C_x - \rho_{yx}\theta C_y\right)\right\}\Bigg]$$

Now, we obtain the minimum MSE of the proposed estimator $\hat{\bar{Y}}_{\alpha,1}$ by minimizing (3.10). Therefore,

The optimum values, γ_1, γ_2 and γ_3 are chosen from the following equations.

$$\frac{\mathfrak{d}\left(MSE\left(\hat{\bar{Y}}_{\alpha,1}\right)\right)}{\mathfrak{d}\gamma_1} = 2\gamma_1\left(1 + \lambda(C_y^2 + 4\theta(\theta C_x^2 - \rho_{yx}C_y)\right) - 2\lambda R_1\gamma_2(\rho_{yx}C_yC_x - 2\theta C_x^2) - 2\lambda R_2\gamma_3C_r(\rho_{yr}C_y + 2\rho_{xr}\theta C_x) - 2\left(1 + \lambda\theta C_x\left(\frac{3}{2}\theta C_x - \rho_{yx}\theta C_y\right)\right)$$

$$\frac{\mathfrak{d}\left(MSE\left(\hat{\bar{Y}}_{\alpha,1}\right)\right)}{\mathfrak{d}\gamma_2} = -2\lambda R_1\gamma_1(\rho_{yx}C_yC_x - 2\theta C_x^2) + \lambda R_1^2\gamma_2C_x^2 + 2\lambda R_1R_2\gamma_3\rho_{xr}C_xC_r - 2R_1\theta C_x^2$$

$$\frac{\mathfrak{d}\left(MSE\left(\hat{\bar{Y}}_{\alpha,1}\right)\right)}{\mathfrak{d}\gamma_3} = -2\lambda R_2\gamma_1C_r(\rho_{yr}C_y + 2\rho_{xr}\theta C_x + 2\lambda R_1R_2\gamma_2\rho_{xr}C_xC_r) - 2\lambda R_1\gamma_3\theta\rho_{xr}C_xC_r$$

Equate the equations to zero and solve simultaneously, the values γ_1, γ_2, and γ_3 are, respectively, given by

$$\gamma_{1(opt)} = \frac{1 - \frac{1}{2}\lambda\theta^2C_x^2}{1 + \lambda C_y^2(1 - R_{y.xr_x}^2)},$$

$$\gamma_{2(opt)} = \frac{\bar{Y}\left[\begin{array}{c}\lambda\theta^3C_x^2(-1+\rho_{xr_x}^2) - C_y\left(1 - \frac{1}{2}\lambda\theta^2C_x^2\right)(\rho_{yx} - \rho_{xr_x}\rho_{yr_x}) \\ +\theta C_x(-1+\rho_{xr_x}^2)[-1+\lambda C_y^2(1-R_{y.xr_x}^2)]\end{array}\right]}{\bar{X}C_x(-1+\rho_{xr_x}^2)[1+\lambda C_y^2(1-R_{y.xr_x}^2)]} \text{ and}$$

$$\pi_{3(opt)} = \frac{\bar{Y}\left(1-\frac{1}{2}\lambda\theta^2 C_x^2\right)C_y\left(\rho_{xr_x}\rho_{yx}-\rho_{yr_x}\right)}{\bar{R}C_{r_x}\left(-1+\rho_{xr_x}^2\right)\left[1+\lambda C_y^2\left(1-R_{y.xr_x}^2\right)\right]}$$

were the coefficient of multiple determination of Y on X and R_x in simple random sampling is defined by $R_{y.xr_x}^2 = \frac{\rho_{yx}^2+\rho_{yr_x}^2-2\rho_{yx}\rho_{yr_x}\rho_{xr_x}}{1-\rho_{xr_x}^2}$.

Substituting γ_1, γ_2 and γ_3 into (3.10) the minimum MSE of the proposed estimator given by

$$MSE_{min}\left(\hat{\hat{Y}}_{\alpha,1}\right) = \frac{\lambda\bar{Y}^2\left[C_y^2\left(1-R_{y.xr_x}^2\right)-\frac{1}{4}\lambda\theta^4C_x^4-\lambda\theta^2C_x^2C_y^2\left(1-R_{y.xr_x}^2\right)\right]}{1+\lambda C_y^2\left(1-R_{y.xr_x}^2\right)} \quad (3.11)$$

A. SECOND PROPOSED ESTIMATOR

In the line of Irfan et al. (2020), the second estimator given by

$$\hat{\hat{Y}}_{\alpha,2} = \frac{1}{2}\gamma_1\left(\frac{\bar{X}}{\bar{x}}+\frac{\bar{x}}{\bar{X}}\right)\hat{\bar{Y}}_{BT,Avg} + \gamma_2(\bar{R}_x-\bar{r}_x) + \gamma_3(\bar{X}-\bar{x})exp\left(\frac{\mathfrak{a}(\bar{X}-\bar{x})}{\mathfrak{a}(\bar{X}+\bar{x})+(\alpha-1)\mathfrak{a}\bar{X}+(\alpha+1)\mathfrak{b}}\right) \quad (3.12)$$

where $\hat{\bar{Y}}_{BT,Avg} = \frac{1}{2}\bar{y}\left(exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)+exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right)$ is the average of the ratio and product estimators and γ_1, γ_2 and γ_3 are suitable chosen constants to be determined.

Rewrite Eq (3.12) in term of relative error, we have

$$\hat{\hat{Y}}_{\alpha,2} = \bar{Y}\gamma_1\left[1+e_0+\frac{5}{8}e_1^2\right]-\gamma_2R_xe_2-\bar{X}\gamma_3e_1\left[1-\theta e_1+\frac{3}{2}\theta^2e_1^2\right] \quad (3.13)$$

As earlier specified $\theta = \frac{a\bar{X}}{(\alpha+1)[\mathfrak{a}\bar{X}+\mathfrak{b}]}$. Thus, the expansion of (3.13), by keeping the e's to second degree and subtracting $\bar{Y}$, given by

$$\hat{\hat{Y}}_{\alpha,2}-\bar{Y} = -\bar{Y}+\bar{Y}\gamma_1+\bar{Y}\gamma_1e_0+\frac{5}{8}\bar{Y}\gamma_1e_1^2-\gamma_2R_xe_2-\bar{X}\gamma_3e_1+\bar{X}\gamma_3\theta e_1^2 \quad (3.14)$$

Taking the expectation of both sides, the bias of the proposed estimator $\hat{\hat{Y}}_{\alpha,2}$ given by

$$Bias\left(\hat{\hat{Y}}_{\alpha,2}\right) \cong \bar{Y}\left(\gamma_1-1+\frac{5}{8}\lambda\gamma_1C_x^2+\lambda\gamma_3R_1\theta C_x^2\right) \quad (3.15)$$

Getting the square of (3.13) under first order approximation, we obtain the MSE of estimator $\hat{\hat{Y}}_{\alpha,2}$ as follows;

$$\left(\hat{\hat{Y}}_{\alpha,2}-\bar{Y}\right)^2 = \bar{Y}^2+\bar{Y}^2\gamma_1^2+\bar{Y}^2\gamma_1^2e_0^2+\frac{5}{4}\bar{Y}^2\gamma_1^2e_1^2+\bar{R}_x^2\gamma_2^2e_2^2+\bar{X}^2\gamma_3^2e_1^2$$
$$-2\bar{Y}\bar{R}_x\gamma_1\gamma_2e_0e_2+2\bar{Y}\bar{X}\gamma_1\gamma_3\theta e_1^2-2\bar{Y}\bar{X}\gamma_1\gamma_3e_0e_1+2\bar{X}\bar{R}_x\gamma_2\gamma_3e_1e_2-2\bar{Y}^2\gamma_1-\frac{5}{4}\bar{Y}^2\gamma_1e_1^2-2\bar{Y}\bar{X}\gamma_3\theta e_1^2$$

$$MSE\left(\hat{\hat{Y}}_{\alpha,2}\right) \cong \bar{Y}^2\left[1+\gamma_1^2\left\{1+\lambda\left(C_y^2+\frac{5}{4}C_x^2\right)\right\}+\lambda R_2^2\gamma_2^2C_r^2-\lambda R_1\gamma_3C_x^2(2\theta+R_1\gamma_3)-2R_2\gamma_1\gamma_2\rho_{yr}C_yC_r-2\lambda R_1\gamma_1\gamma_3\left(\rho_{yx}C_yC_x-\theta C_x^2\right)+2\lambda R_1R_1\gamma_2\gamma_3\rho_{xr}C_xC_r-2\gamma_1\left(1+\frac{5}{8}\lambda C_x^2\right)\right] \quad (3.16)$$

Here, we also obtain the minimum MSE of the proposed estimator $\hat{\hat{Y}}_{\alpha,2}$ by minimizing (3.16). Therefore, the optimum values, γ_1, γ_2 and γ_3 are chosen from the following equations.

$$\frac{\partial\left(MSE\left(\hat{\hat{Y}}_{\alpha,2}\right)\right)}{\partial\gamma_1} = 2\gamma_1\left(1+\lambda(C_y^2+\frac{5}{4}C_x^2)\right)-2\lambda R_2\gamma_2\rho_{yr}C_yC_x-2\lambda R_1\gamma_3(\rho_{yx}C_yC_x-\theta C_x^2)-2\left(1+\frac{5}{8}\lambda C_x^2\right)$$

$$\frac{\partial\left(MSE\left(\hat{\hat{Y}}_{\alpha,2}\right)\right)}{\partial\gamma_2} = -2\lambda R_2\gamma_1\rho_{yr}C_yC_r+\lambda R_2^2\gamma_2C_r^2+2\lambda R_1R_2\gamma_3\rho_{xr}C_xC_r$$

$$\frac{\partial\left(MSE\left(\hat{\hat{Y}}_{\alpha,2}\right)\right)}{\partial\gamma_3} = -2\lambda R_1\gamma_1(\rho_{yx}C_yC_x-\theta C_x^2)+2\lambda R_1R_2\gamma_2\rho_{xr}C_xC_r+\lambda R_1^2\gamma_3C_x^2-2\lambda R_1\theta C_x^2$$

Equate the equations to zero and solve simultaneously, the values γ_1, γ_2, and γ_3 are, respectively, given by

$$\gamma_1 = \frac{\left(1+\frac{5}{8}\lambda C_x^2\right)(1-\rho_{xr}^2)+\theta C_yC_x(\rho_{yx}-\rho_{yr}\rho_{xr})-\lambda\theta^2C_x^2}{\left\{1+\frac{5}{4}\lambda C_x^2+\lambda C_y^2(1-R_{y.xr}^2)\right\}(1-\rho_{xr}^2)+2\lambda\theta C_yC_x(\rho_{yx}-\rho_{yr}\rho_{xr})-\lambda\theta^2C_x^2}$$

$$\gamma_2 = \frac{\rho_{yr}C_y\left\{\left(1+\frac{5}{8}\lambda C_x^2\right)(1-\rho_{xr}^2)+\theta C_yC_x(\rho_{yx}-\rho_{yr}\rho_{xr})-\lambda\theta^2C_x^2\right\}-\rho_{xr}\left\{\frac{5}{8}\lambda\theta C_x^3+\lambda\theta C_y^2C_x(1-\rho_{yr}^2)+\left(1+\frac{5}{8}\lambda C_x^2\right)C_y(\rho_{yx}-\rho_{yr}\rho_{xr})\right\}}{R_2C_r\left[\left\{1+\frac{5}{4}\lambda C_x^2+\lambda C_y^2(1-R_{y.xr}^2)\right\}(1-\rho_{xr}^2)+2\lambda\theta C_yC_x(\rho_{yx}-\rho_{yr}\rho_{xr})-\lambda\theta^2C_x^2\right]}$$ and

$$\gamma_3 = \frac{\theta C_x\left\{\frac{5}{8}\lambda C_x^2+\lambda C_y^2(1-\rho_{yr}^2)\right\}+\left(1+\frac{5}{8}\lambda C_x^2\right)C_y(\rho_{yx}-\rho_{yr}\rho_{xr})}{R_1C_x\left[\left\{1+\frac{5}{4}\lambda C_x^2+\lambda C_y^2(1-R_{y.xr}^2)\right\}(1-\rho_{xr}^2)+2\lambda\theta C_yC_x(\rho_{yx}-\rho_{yr}\rho_{xr})-\lambda\theta^2C_x^2\right]}$$

Also, the coefficient of multiple determination of Y on X and R_x in simple random sampling is defined by $R_{y.xr_x}^2 = \frac{\rho_{yx}^2+\rho_{yr_x}^2-2\rho_{yx}\rho_{yr_x}\rho_{xr_x}}{1-\rho_{xr_x}^2}$.

Substituting γ_1, γ_2 and γ_3 into (3.16) the minimum MSE of the proposed estimator given by

$$MSE_{min}\left(\hat{\hat{Y}}_{\alpha,2}\right) = \frac{\bar{Y}^2\left[\left\{\lambda C_y^2(1-R_{y.xr}^2)-\frac{25}{64}\lambda^2\right\}(1-\rho_{xr}^2)-\lambda\theta C_x^2\left\{\lambda\theta C_y^2(1-\rho_{yr}^2)+\frac{5}{4}\lambda C_yC_x(\rho_{yx}-\rho_{yr}\rho_{xr})\right\}\right]}{\left\{1+\frac{5}{4}\lambda C_x^2+\lambda C_y^2(1-R_{y.xr}^2)\right\}(1-\rho_{xr}^2)+2\lambda\theta C_yC_x(\rho_{yx}-\rho_{yr}\rho_{xr})-\lambda\theta^2C_x^2} \quad (3.17)$$

IV. APPLICATIONS

In this section, we employed simulation study and real datasets to evaluate the performance of the proposed estimators and compared to the existing estimators in terms of mean squared errors (MSEs) and percentage relative efficiencies (PREs). The expressions in section 2 and 3 are used to compute the MSEs and the PREs of the estimators relating to the mean per unit estimator $\bar{Y}$ given by

$$PRE = \frac{var(\bar{y})}{MSE(*)} \quad (4.1)$$

A. SIMULATION STUDY

Here, we conducted simulation study to evaluate the performance of the proposed estimators and compare their efficiency to the existing estimators. Two sub-population of size 1000 were simulated from multivariate normal population with different covariance matrix to present the distribution of the study variable Y, auxiliary variable X and the rank of the auxiliary variable R_x. The theoretical mean μ and the covariance matrix Σ of the simulated populations are as $\mu = [5, 5,5]$ and population I:

$$\Sigma = \begin{bmatrix} 4 & 2.7 & 2.2 \\ 2.7 & 5 & 3 \\ 2.2 & 3 & 6.75 \end{bmatrix} \begin{matrix} \rho_{yx} = 0.5628 \\ \rho_{yr} = 0.4252 \\ \rho_{xr} = 0.5405 \end{matrix},$$

Population II:

$$\Sigma = \begin{bmatrix} 10 & 8.0 & 8.1 \\ 8.0 & 8.5 & 8.2 \\ 8.1 & 8.2 & 8.4 \end{bmatrix} \begin{matrix} \rho_{yx} = 0.8745 \\ \rho_{yr} = 0.8917 \\ \rho_{xr} = 0.9683 \end{matrix} \text{ respectively.}$$

The population I has weak correlation while the population II has strong correlation. Table 4.1 to 4.4 present the MSE values of the proposed estimators for different sample size $n = 10, 20, 50$ and 100 based on two different simulated populations. The MSE values of the first proposed estimators $\hat{\bar{Y}}_{\alpha,1}$ for different choice of α between 0.1 and 1 are reported in Table 4.1 and 4.3 for simulated population I and II, respectively. and Table 4.2 and 4.4 are for the second proposed estimators $\hat{\bar{Y}}_{\alpha,2}$.

(Insert Tables 4.1 to 4.4)

For notational convenience, the estimators developed by Haq et al. (2017) and Irfan et al. (2020) are referred to as the estimators $\hat{\bar{Y}}_{1,1}$ and $\hat{\bar{Y}}_{1,2}$, respectively. It is clear that there are similarities in the MSE values of the proposed estimators from the various choices of α considered in this study. The results in Table 4.1 to 4.4 are summarized as follows:

- ✓ It is noticed that the estimators designed based on choice of α between 0.1 and 0.9 produce very smaller MSEs over any other class of estimators i.e., $\hat{\bar{Y}}_{1,1}$ and $\hat{\bar{Y}}_{1,2}$,
- ✓ The choice of smaller α enhances the performance of the estimators especially, when the sample size is smaller.
- ✓ It is noticed from the results that the MSE values of all classes of estimators $\hat{\bar{Y}}_{\alpha,1}$ and $\hat{\bar{Y}}_{\alpha,2}$ are become significantly smaller with an increase in sample size n. Thus, increasing in sample size enhances the efficiency of the estimators.
- ✓ It is equally noticed that there is hardly any difference in the MSEs of the proposed estimators when the sample size is increasing.
- ✓ The results indicate substantial improvement when estimators $\hat{\bar{Y}}_{0.1,1}$ and $\hat{\bar{Y}}_{0.1,2}$, are applied to study and auxiliary variables with correlation.

B. REAL DATASETS

In this section, we use three different population datasets. The first population is related to the tube well and net irrigated area (in hectares) for 69 villages of Dorah development block Punjab, India (Singh and Mangat, 1996). Tube well is the study variable while the irrigated area is the auxiliary variable. The second population data is on Primary and Secondary schools for 923 districts of Turkey (Ministry of Education, Republic of Turkey, 2007). Taking the number of Teachers and Students as study and auxiliary variables, respectively. The last Population data is on Apple production in 854 villages of Turkey (Institute of Statistics, Republic of Turkey, 1999). The Amount and the number of apple trees are taking as study and auxiliary variables, respectively. These population datasets are recently considered in the work of Irfan et al. (2020). Their values and parameters are presented in Table 4.5.

Parameters	Population I	Population II	Population III
N	69	924	854
n	12	180	290
$\bar{Y}$	135.2608	436.4345	2930.12
$\bar{X}$	345.7536	1144.5	37600.11
$\bar{R}_x$	34.9565	461.9642	426.87.47
C_y	0.8422	1.7185	5.8379
C_x	0.8479	1.8645	3.8509
C_r	0.5747	0.577	0.1883
ρ_{yx}	0.9224	0.9543	0.9165
ρ_{yr}	0.72159	0.6442	0.2585
ρ_{xr}	0.8185	0.6306	0.3458
$\beta_2(x)$	7.2159	18.7208	312.6051

Table 4.5: Values and Parameters of the Population datasets

In order to determine the performance of the proposed estimators over real-life situations. The mean square errors (MSEs) and percentage relative efficiencies (PREs) of the proposed estimators were obtained using population datasets reported in Table 4.5. As a matter of fact, the proposed estimators, are special case of the existing difference-ratio-type exponential estimator $\hat{\bar{Y}}_H$ or $\hat{\bar{Y}}_{1,1}$ (Haq et al.,2017) when $\alpha = 1$ and estimator $\hat{\bar{Y}}_I$ or $\hat{\bar{Y}}_{1,2}$ (Irfan et al., 2020) when $\alpha = \mathfrak{a} = 1$ *and* $\mathfrak{b} = 0$, respectively. Following the implementation of these estimators on different population datasets, the MSE and PRE values are computed and presented in Table 4.6 to 4.11

(Insert Table 4.6 to 11)

According to these results, it is noticed that the MSE and PRE of the estimators are decreasing and increasing, respectively as the choice of α value reduces when the class of estimator kept constant. As a result of this, the class of estimators $\hat{\bar{Y}}_{0.1,1}$ and $\hat{\bar{Y}}_{0.1,2}$, provided the smallest MSE and highest PRE values, respectively. In the same vein, it can be observed that the class of estimator $\hat{\bar{Y}}_{0.1,2}$ has maximum gain of PRE over all other estimators considered for the three population datasets, when constant $\mathfrak{a}$ and $\mathfrak{b}$ are chosen to be 1 and 0, respectively.

V. CONCLUSION

In this study, we have proposed some new difference-ratio-type exponential estimators of the finite population mean under simple random sampling without replacement. The study uses supplementary information on both the actual auxiliary variable and the rank of auxiliary variable. We derived the mathematical properties of the proposed estimators up to first order approximation including the bias, MSEs and the minimum MSEs. Based on the simulated and real population datasets, we compute the MSEs and PREs for the proposed difference-ratio-type exponential estimators. According to the work of Irfan et al. (2020), existing difference-ratio-type exponential estimator $\hat{\bar{Y}}_I$ or $\hat{\bar{Y}}_{1,2}$ for population mean under simple random sampling has been shown to be most efficient in all the estimators considered in their study. Nevertheless, the MSE and PRE of our proposed estimators (i.e $\hat{\bar{Y}}_{\alpha,2}$, $\alpha = 0.1, \dots, 0.9$) for various classes are smaller and higher than the MSE and PRE of the existing estimator $\hat{\bar{Y}}_I$, respectively. This means that, the proposed estimators $\hat{\bar{Y}}_{\alpha,2}, \alpha = 0.1, \dots 0.9$ gained higher precision over other existing estimator $\hat{\bar{Y}}_I$ and other counterparts. Particularly, class of estimator $\hat{\bar{Y}}_{0.1,2}$.

REFERENCES

[1] Haq, A., Khan, M., and Hussain, Z (2017) A new estimator of finite population mean based on the dual use of the auxiliary information", Communications in Statistics Theory and Methods, 46(9), pp. 4425-4436.

[2] Kadilar, C. and Cingi, H (2004). Ratio estimators in simple random sampling", Applied Mathematics and Computation, 151, pp. 893 -902

[3] Kadilar, C. and Cingi, H (2006a). Improvement in estimating the population mean in simple random sampling", Applied Mathematics Letters, 19, pp. 75-79.

[4] Singh, H.P. and Solanki, R.S (2013). An efficient class of estimators for the population mean using auxiliary information", Communications in Statistics-Theory and Methods, 42, pp. 145-163.

[5] Shabbir, J., Haq, A., and Gupta, S. (2014). A new difference cum-exponential type estimator of finite population mean in simple random sampling", Revista Colombiana de Estadistica, 37(1), pp. 199-211.

[6] Singh, R. and Mangat, N.S (1996). Elements of survey sampling", Norwell, MA: Kluwer Academic Publishers.

[7] Cochran W. G. (1940). The estimation of the yields of cereal experiments by sampling for the ratio gain to total produce. Journal of Agriculture Society 30:262-275.

[8] Murthy M. N. (1964). Product method of estimation. Sankhyā: The Indian Journal of Statistics, Series A (1961-2002) 26(1):69-74.

[9] Bahl S., Tuteja, R. K. (1991). Ratio and product type exponential estimators. Journal of Information and Optimization Sciences 12(1):159-164.

[10] Bedi P. K. (1996). Efficient utilization of auxiliary information at estimation stage. Biometrical Journal 38(8):973-976.

[11] Singh G. N. (2003a). On the improvement of product method of estimation in sample surveys. Journal of the Indian Society of Agricultural Statistics 56(3):267-275.

[12] Singh H. P., Tailor, R. (2003). Use of known correlation coefficient in estimating the finite population mean. Statistics in Transition 6(4):555-560.

[13] Singh H. P., Solanki, R. S. (2013). An efficient class of estimators for the population mean using auxiliary information. Communications in Statistics - Theory and Methods 42(1):145- 163.

[14] Shabbir J., Gupta, S. (2010). On estimating finite population mean in simple and stratified random sampling. Communications in Statistics - Theory and Methods 40(2):199-212.

[15] Khoshnevisan M., Singh, R., Chauhan, P., Sawan, N., Smarandache, F. (2007). A general family of estimators for estimating population mean using known value of some population parameter(s). Far East Journal of Theoretical Statistics 22(9):181-191.

[16] Singh R., Chauhan, P., Sawan, N., Smarandache, F. (2009). Improvement in estimating the population mean using exponential estimator in simple random sampling. International Journal of Statistics & Economics 3(A09):13-18.

[17] Grover L. K., Kaur, P. (2011). An improved estimator of the finite population mean in simple random sampling. Model Assisted Statistics and Applications 6(1):47-55.

[18] Grover L. K., Kaur, P. (2014). A generalized class of ratio type exponential estimators of population mean under linear transformation of auxiliary variable. Communications in Statistics - Simulation and Computation 43(7):1552-1574

[19] Irfana, M , Javeda, M. and Haider Bhattia S. (2022) Difference-type-exponential estimators based on dual auxiliary information under simple random sampling. Scientia Iranica E 29(1).