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A Note On Representation Of Dynkin Quiver Of Type A4

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***Abstract:** In this paper we investigate Gabriel's theorem on representation of Dynkin quivers of type A_4. We show that there is a correspondence between the root system and the indecomposable representations of quivers.*

***Keywords:** Gabriel's theorem, Dynkin quivers, graphs, isomorphism, path algebra.*

I. INTRODUCTION

Throughout this paper, we will work over the field of complex numbers. A quiver is a finite directed graph. A representation of a quiver is an assignment of a vector space to each vertex and a linear map to each arrow. The main objective is to illustrate Gabriel's theorem characterizing representations of quivers of finite type. Gabriel's theorem implies that a quiver is of finite representation type if and only if the undirected graph obtained is Dynkin. Moreover, the isomorphism classes of indecomposable representation correspond to the positive roots of the associated root system.

A. REPRESENTATION OF THE DYNKIN QUIVER

In what follows, a graph means a connected graph whose vertices are finite with no loops.

DEFINITION 1.1.1: A quiver Q is a directed graph; for example $Q = \cdot\!\rightarrow\!\cdot\!\rightarrow\!\cdot\!\rightarrow\!\cdot$

DEFINITION 1.1.2: A representation V of a quiver Q is said to be indecomposable if it is non-zero and it is not isomorphic to a direct sum of two non-zero representation. This can be written as $v \ncong V_1 \oplus V_2$ where $V_i \neq \{0\}$ for $i = 1, 2$.

DEFINITION 1.1.3: A quiver has finite representation type if it has only finitely many indecomposable representations up to isomorphism.

DEFINITION 1.1.4: An integral quadratic form in variables is $q_1 = \sum_{i=1}^{4} x^2{}_i - \sum_{i\cdot-\cdot j} a_{ij} x_i x_j$, $a_{ij} \in \mathbb{Z}$. This is a homogeneous polynomial of degree 2. Let Γ be a graph, the quadratic for of $q_\Gamma = \sum_{i=1}^{n} x^2{}_i - \sum_{i\cdot-\cdot j} a_{ij} x_i x_j$, $a_{ij} \in \mathbb{Z}$.

Let $q_\Gamma : \mathbb{Z}^4 \longrightarrow \mathbb{Z}$, $x = (x_1, \ldots, x_4) \in \mathbb{Z}^4$ is called a root of q_Γ if $q_\Gamma(x) = 1$.

We say that $x = (x_1, \ldots, x_4)$ is positive if $(X) \neq \{0\}$ and $x_i \geq 0$ for all $1 \leq i \leq 4$.

II. MAIN RESULT

THEOREM 2.1

A quiver has finite representation if and only if the undirected graph of a quiver is Dynkin. Moreover, Gabriel established a correspondence between the root system and the indecomposable representation of the quiver.

ILLUSTRATION

Consider the following quiver Q_1 and Q_2 where $Q_1 = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$ and $Q_2 = 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xrightarrow{\gamma} 4$.

The quadratic forms of the above quivers are:

$q_{Q_1}(x) = \sum_{i=1}^{4} x^2{}_i - \sum_{i-j} a_{ij}x_ix_j = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 - x_1x_2 - x_2x_3 - x_3x_4$ and

$q_{Q_2}(x) = \sum_{i=1}^{4} x^2{}_i - \sum_{i-j} a_{ij}x_ix_j = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 - x_1x_2 - x_2x_3 - x_3x_4 = x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 - x_1x_2 - x_2x_3 - x_3x_4$,

respectively. We notice that Q_1 and Q_2 have the same quadratic form. The simple positive roots of Q_1 are (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (0,1,1,0), (0,0,1,1), (1,1,1,0), (0,1,1,1) and (1,1,1,1). Since Q_1 and Q_2 have the same quadratic form, they have the same roots.

REMARK 2.1: Note that the number of positive roots of two quivers with non-isomorphic paths algebra does not depend on the orientation.

The indecomposable representations of Q_1 *are*:

$V_{1000}: \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0$

$V_{0100}: 0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0$

$V_{0010}: 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0$

$V_{0001}: 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C}$

$V_{1100}: \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0$

$V_{0110}: 0 \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0$

$V_{0011}: 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C}$

$V_{1110}: \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0$

$V_{0111}: 0 \xrightarrow{0} \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}$

$V_{1111}: \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}$

By Gabriel's theorem we notice that, the number of positive roots of Q_1 corresponds to the number of indecomposable representation of Q_1. Thus there is a correspondence between the root system and the indecomposable representation of Q_1. The indecomposable representation of Q_2 will be similar to those of Q_1 since they have the same root.

REMARK 2.2: We notice that, for Dynkin quiver, the number of indecomposable representation of the two quivers with non-isomorphic path algebras does not depend on their orientation.

DEFINITION 2.1: Let Q be a quiver with any labelling 1,2.., n of the vertices. Let $V = v_1, \dots, v_n$ be a representation of Q. we call $\dim(V) = \dim v_1, \dots, \dim v_n$ the dimension vector of the representation. In the above, the dimension vector of the irreducible representation of Q_1 are (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (0,1,1,0), (0,0,1,1), (1,1,1,0), (0,1,1,1) and (1,1,1,1) which are the same as that of Q_2.

REMARK 2.3: The dimension vector of the indecomposable representation of two quivers with non-isomorphic path algebras does not depend on their orientation.

DEFINITION 2.2: A path in a quiver Q is a sequence $P = \alpha_1\alpha_2, \dots, \alpha_l$ of l arrows in Q such that $t(\alpha_1) = S(\alpha_{1+1})$ for all $1 \leq i \leq l$, where $t(-)$ and $s(-)$ denote the target and the source of the arrow respectively. We then say p has length, l. At each vertex i, we have length zero paths denoted by e_i. For example, $Q = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4$. The path of Q_1 is $P = \alpha_1\alpha_2\alpha_3$. The length of $p = 3$ and the length zero paths are e_1, e_2, e_3 and e_4.

DEFINITION 2.3: For a quiver Q, the path algebra $\mathbb{C}\,Q$ is the vector space with basis the paths of Q. the multiplication is defined as;

$$p.q = \begin{cases} pq, & t(p) = s(q) \\ 0, & t(p) \neq s(q) \end{cases}$$

For $Q_1 = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$, the basis of $\mathbb{C}\,Q_1$ will be $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma\}$. Thus $\mathbb{C}Q_1$ has dimension 10.

$Q_2 = 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xrightarrow{\gamma} 4$, the basis of $\mathbb{C}\,Q_2$ will be $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma\}$. The dimension of $\mathbb{C}\,Q_2$ is 7.

REMARK 2.4: We notice that the dimension of the path of the path algebra $\mathbb{C}\,Q$ depends on the orientation of the quiver.

DEFINITION 2.4: Let Q be a quiver, the opposite quiver denoted by Q^{op} is the quiver with same vertices as Q and an arrow $\alpha^*: j \to i$ for each arrow $\alpha: i \to j$. Using the quiver above, the opposite quiver of Q_1 and Q_2 and $Q_1{}^{OP} = 1 \xleftarrow{\alpha^*} 2 \xleftarrow{\beta^*} 3 \xleftarrow{\gamma^*} 4$ and $Q_2{}^{OP} = 1 \xleftarrow{\alpha^*} 2 \xrightarrow{\beta*} 3 \xleftarrow{\gamma^*} 4$ resptively.

DEFINITION 2.5: An A-module M is simple if it is not the zero module and there are no other proper sub-modules. Let M be an A-module. A composition series of M is a chain $0 = M_0 \subseteq M_1 \subseteq \dots M_t = M$ of submodules such that all the factors M_i/M_{i-1} with $1 \leq i \leq t$ are simple. The number t is called the length of the composition series. Now consider the $\mathbb{C}\,Q^{OP}$ modules associated to the representation, in order to do this we need to have a basis of the module associated to each representation and the action table of the basis of the quiver against the basis of the module.

For instance, using $\mathbb{C}Q_1{}^{OP} = 1 \xleftarrow{\alpha^*} 2 \xleftarrow{\beta^*} 3 \xleftarrow{\gamma^*} 4$ for the representation $V_{1000}: \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0$, let M_{1000} have the basis (X), since the basis of 0 is the empty set, the action table is,

	x
e_1	x
e_2	0
e_3	0
e_4	0
α^*	0
β^*	0
γ^*	0
$\beta^*\alpha^*$	0
$\gamma^*\beta^*$	0
$\gamma^*\beta^*\alpha^*$	0

which is equal to S_1 where S_1 is simple (1-dimensional).

$V_{0100}: 0 \xrightarrow{0} \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0$, basis of M_{0100} is $\{y\}$ which is equal to S_2 *yields,*

	y
e_1	0
e_2	y
e_3	0
e_4	S_2 =0
α^*	0
β^*	0
γ^*	0
$\beta^*\alpha^*$	0
$\gamma^*\beta^*$	0
$\gamma^*\beta^*\alpha^*$	0

$V_{1100}: \mathbb{C} \xrightarrow{\{x\}} \mathbb{C} \xrightarrow{\{y\}} 0 \xrightarrow{0} 0$ where basis of M_{1100} is {*x*,*y*}

	x	y
e_1	x	0
e_2	0	y
e_3	0	0
e_4	0	0
α^*	y	0
β^*	0	0
γ^*	0	0
$\beta^*\alpha^*$	0	0
$\gamma^*\beta^*$	0	0
$\gamma^*\beta^*\alpha^*$	0	0

Let L =

	y
e_1	0
e_2	y
e_3	0
e_4	0
α^*	0
β^*	0
γ^*	0
$\beta^*\alpha^*$	0
$\gamma^*\beta^*$	0
$\gamma^*\beta^*\alpha^*$	0

L is a sub-module of M_{1100}, therefore $L \cong S_2$. Taking the quotient of M_{1100}/L we obtain

	X+L
e_1	X+L
e_2	0+L
e_3	0+L
e_4	0+L
α^*	0+L
β^*	0+L
γ^*	0+L
$\beta^*\alpha^*$	0+L
$\gamma^*\beta^*$	0+L
$\gamma^*\beta^*\alpha^*$	0+ L

which isisomorphic to S_1. The composition series associated to this representation is $O \subset L \subset M_{1100}$. A similar approach is used to obtain the composition series of the $\mathbb{C}Q_1{}^{OP}$- modules associated to the remaining representations. The same argument is applied in obtaining composition series of the $\mathbb{C}Q_2{}^{OP}$- modules.

III. CONCLUSION

We observe that the number of indecomposable representations of Q_1 and Q_2 and their dimensional vectors does not depend on the orientation which is the generalization of the Gabriel's theorem. In addition, there is a correspondence between the root system and the indecomposable representation of the quiver.

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