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# The Split Decompositions Of Finite Separable Metacyclic 2-Group

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***Abstract:** Given a finite separable metacyclic 2-group G, it is always possible to express G as a semidirect product of a cyclic group with another cyclic group. In this paper, we implement the use of Group Application Package (GAP) Software to determine the split decompositions of a finite separable metacyclic 2-group up to isomorphism, where the dihedral group $D_{16}$ of order $2^5$ and its presentations was derived and shown to be separable. The finite groups were generated and expressed as the semidirect product of cyclic subgroups.*

***Keywords:** Complement, Metacyclic group, Dihedral group; Separable, Semidirect product, Split decompositions*



## I. INTRODUCTION

A subgroup $N$ of a group $G$ is complemented in $G$ if there exists another subgroup $H$ of $G$ such that $G = NH$ and whenever $x \in N$ and $x \in H$, then $x = 1$, the identity element of $G$. i.e. $N \cap H = \{1\}$. If in addition, $N \lhd G$, then $G$ is said to split over $N$ and is written as $G = [N]H$. In this case, we say that $G$ is the semidirect product of its subgroups $N$ and $H$. If further, $G = [N]H = [N_1]H_1$, with $N \cong N_1$ and $H \cong H_1$, then the two split decompositions $[N]H$ and $[N_1]H_1$ are said to be isomorphic. A group is said to be separable if it splits over a nontrivial proper normal subgroup and inseparable otherwise.

Any metacyclic $p$-group can be presented by the relation $G = \langle x, y \mid x^{p^n} = 1, y^{p^m} = x^t, x^y = x^r \rangle$ [1]. If G is separable, then by the same result, we can make $t$ to be 0 so that $G = \langle x, y \mid x^{p^n} = y^{p^m} = 1, x^y = x^r \rangle$. In this case, $G = [\langle x \rangle]\langle y \rangle$ and in our result, we have shown that the separable metacyclic $p$-groups with $p$ odd have split decompositions isomorphic to $G$. However, this is not the case for all metacyclic 2-groups, particularly for Dihedral groups $D_n$ of order $2n$. For instance, consider the dihedral group $D_4 = \langle \alpha, \beta \mid \alpha^4 = \beta^2 = 1, \alpha^\beta = \alpha^3 \rangle$ of order 8. While $D_4 = [\langle \alpha \rangle]\langle \beta \rangle$, it has a non-isomorphic decomposition, where $D_4 = [\langle \alpha^2, \beta \rangle]\langle \alpha\beta \rangle$, with $\langle \alpha^2, \beta \rangle$ isomorphic to the Klein 4-group. For general presentation and more information, see [2].

This paper is a review of the work of Kirtland, [10]. Our aim is to study the finite separable metacyclic 2-group, the necessary and sufficient conditions under which a separable metacyclic 2-group has all its split decompositions isomorphic as discussed by Kirtland and then implement the use of GAP to determine its splits decompositions. The concept of metacyclic 2-groups and metacyclic $p$-groups in general, have been extensively studied by many authors. For more information and results, see the work of Brandle and Verardi [3], Lie dhal [4], King [1], Beuerle [5], Hempel [6], and Sire [7]. In particular, separable metacyclic groups have been studied by Sim [8] and Jackson [9]. However, the simple condition for which a separable metacyclic 2-group has all its split decompositions isomorphic has not been directly addressed by these authors.

We used standard notations and for a $p$-group $G$, the subgroup $\Omega_i$ of $G$ is given by $\Omega_i(G) = \langle g \mid g \in G, g^{p^i} = 1 \rangle$. The center of a group $G$ is denoted by $Z(G)$ and $\Phi(G)$ will denote its Frattini subgroup. Our research is limited to finite groups.

We shall see from the following theorem that if $G$ is a metacyclic $p$-group for $p$ an odd prime, then all its split decompositions are isomorphic.

*THEOREM 1.1:* Given a separable metacyclic *p*-group *G* for *p* an odd prime, then all its split decompositions are isomorphic [10].

*PROOF.* By [1], we have $G = \langle x, y \mid x^{p^n} = y^{p^m} = 1, x^y = x^r \rangle$ and so, $G = [\langle x\rangle]\langle y\rangle$. Let $G = [N]H$ be another split decomposition for *G*. If *G* is Abelian, then $G = \langle x\rangle \times \langle y\rangle = N \times H$ and without any loss of generality, we get by [10], that $N \cong \langle x\rangle$ and $H \cong \langle y\rangle$.

Assume *G* is not Abelian. Then by [12], we have $|\Omega_1(G)| = p^2$ and since both $\Omega_1(N)$ and $\Omega_1(H)$ are nontrivial and both contained in $\Omega_1(G)$, it follows that $|\Omega_1(N)| = |\Omega_1(H)| = p$. But $p > 2$. Hence *N* and *H* are cyclic by [5] and by [7], $N \cong \langle x\rangle$ and $H \cong \langle y\rangle$.*

## II. SEPARABLE METACYCLIC 2-GROUPS; M = 1

In this section, split decompositions of metacyclic 2-groups *G* of the form $G = \langle x, y \mid x^{2^n} = y^2 = 1, x^y = x^r \rangle$ are investigated. Kirtland [10] in his paper also investigated the split decompositions of metacyclic 2-group. The novelty in this paper is to show how to investigate the split decompositions of a finite metacyclic 2-group using GAP Software and to determine whether the decompositions are isomorphic. If *G* is abelian, then obviously, all its split decompositions are isomorphic by [11]. Otherwise, if $n = 2$, then *G* is the dihedral group $D_8$ and not all of its split decompositions are isomorphic [10]. The case $n \geq 3$ shall be considered in the next section where *G* is non-Abelian.

We shall now consider the following theorem as stated by Kirtland [10], and then implement it in GAP by generating a Dihedral group $D_{16}$ of order 32 and its presentations.

*THEOREM 2.1:* Let *G* be any non-Abelian metacyclic 2-group with presentation $G = \langle x, y \mid x^{2^n} = y^2 = 1, x^y = x^r \rangle$ with $n \geq 3$. Then the split decompositions of *G* are isomorphic if and only if $r = 2^{n-1} + 1$.

*PROOF:* Assume that all split decompositions of *G* are isomorphic. Now consider the normal subgroup $N = \langle x^2, xy\rangle$ of *G*. Then since *N* is complemented in *G*, we have $G = [\langle x^2, xy\rangle]\langle y\rangle$, $\langle x^2, xy\rangle \cong \langle x\rangle$ and $\langle x^2, xy\rangle$ is Abelian. Thus $x^2 = (x^2)^{xy} = (x^y)^2 = (x^r)^2 = x^{2r}$ and $2r \equiv 2 \ (mod\ 2^n)$. Hence, $r \equiv 1\ (mod\ 2^{n-1})$ or $r = 2^{n-1} + 1$.

Conversely, suppose $r = 2^{n-1} + 1$ and that $G = [N]H$. Then obviously, $(x^2)^y = (x^y)^2 = (x^r)^2 = x^{2r} = x^2$ and $\langle x^2\rangle = Z(G)$. Furthermore, $N \cap Z(G) \neq \{1\}$. This implies that $\langle x^{2^{n-1}}\rangle \in N$ and $x^{2^{n-1}} \notin H$. If in addition, $[x, y] = x^{-1}x^y = x^{-1+r} = x^{2^{n-1}}$, then $G' = \langle x^{2^{n-1}}\rangle$.

Next, let $\overline{G} = G/\langle x^{2^{n-1}}\rangle \cong Z_{2^{n-1}} \times Z_2$. Then consequently, $\overline{G} = N/\langle x^{2^{n-1}}\rangle \times H\langle x^{2^{n-1}}\rangle/\langle x^{2^{n-1}}\rangle = \overline{N} \times \overline{H}$ with $\overline{N} \neq \{1\}$ and $\overline{H} \cong H$. Hence, we have by [11], that $\overline{N} \cong Z_{2^{n-1}}$ and $\overline{H} \cong H \cong Z_2$ or $\overline{N} \cong Z_2$ and $\overline{H} \cong H \cong Z_{2^{n-1}}$.

Finally, consider the case $\overline{N} \cong Z_2$ and $\overline{H} \cong H \cong Z_{2^{n-1}}$. Then we have $|N| = 4$ and $|H/C_H(N)| \leq |Aut(N)|_2 = 2$. But if *H* is cyclic, then we have $C_H(N) \leq H \cap Z(G) = \{1\}$ and $|H| \leq 2$. Thus $|G| = 8$ and $n = 2$. But this contradicts the fact that $n \geq 3$. Consequently, $\overline{N} \cong Z_{2^{n-1}}$ and $H \cong \overline{H} \cong Z_2 \cong \langle b\rangle$. Hence, *N* is a maximal subgroup of *G*.

Now since $G/\langle x^2\rangle = Z_2 \times Z_2$ and $\Phi(G) = \langle x^2\rangle$, the only possible maximal subgroups of *G* are $\langle x\rangle$, $\langle x^2, y\rangle$, and $\langle x^2, xy\rangle$. But $(xy)^2 = xx^y = x^{r+1} = x^{2^{n-1}+2}$ and $|x^{2^{n-1}+2}| = 2^{n-1}$. Hence, $|xy| = 2^n$ and $\langle x^2, xy\rangle \cong \langle xy\rangle \cong \langle x\rangle$. Now consider the subgroup $\langle x^2, y\rangle$. Since $Z(G) = \langle x^2\rangle$, $|\Omega_1(G)| = 4$ and $\Omega_1(G) = \langle x^{2^{n-1}}, y\rangle$, we have $\Omega_1(G) \leq \langle x^2, y\rangle$. Furthermore, $\Omega_1(H) \leq \Omega_1(G) \leq \langle x^2, y\rangle$. Therefore if $N = \langle x^2, y\rangle$, then $N \cap H \neq \{1\}$, a contradiction. Hence, $N \cong \langle x\rangle$, $H \cong \langle y\rangle$ and all split decompositions are isomorphic.*

*REMARK 2.2:* Consider the dihedral group $D_{16}$, of order $2^5$ with the presentation

$$D_{16} = \langle \alpha, \beta \mid \alpha^{2^n} = \beta^2 = 1, \alpha^\beta = \alpha^r \rangle.$$

We obtained the following results from GAP.

```
gap> G:= DihedralGroup(IsPermGroup, 32);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16),
(2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10) ])
gap> r:= (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16);;
gap> N:= Subgroup(G, [r]);;
gap> f:= (2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10);;
gap> H:= Subgroup(G, [f]);;
gap> Size(N); Size(H);
16
2
gap> IsCyclic(N);
true
gap> IsCyclic(H);
true
gap> IsNormal(G, N);
true
gap> IsNormal(G, H);
false
gap> D:= DirectProduct(N, H);;
gap> Size(D);
32
gap> Center(G);
Group([ (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) ])
gap> s:= (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16);;
gap> Z:= Subgroup(G, [s]);;
gap> Size(Z);
2
gap> IsNormal(G, Z);
true
```

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```
gap> IsCyclic(Z);
true
gap> quit
```

Here, the subgroup $N$ of order 16 consists of all the rotations in $D_{16}$ as follows:

If we let $\alpha$ = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16), then $N$ is given by the following presentation

$N = \{\alpha^n : 1 \le n \le 16$, where $\alpha^{16} = \alpha^0 = 1\}$.

The cyclic subgroup $N$ of index 2 in $D_{16}$ is normal as shown above. It is also true by Lagrange's theorem since $[D_{16} : N] = |D_{16}|/|N| = 2$.

The subgroup H of order 2 is the cyclic subgroup of $D_{16}$ generated by the reflection $\beta$ = (2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10) along the line of symmetry (1,9), where $H$ is given by the presentation

$$H = \langle\beta\rangle = \{\beta^n : 1 \le n \le 2\} \quad \text{or}$$

$$H = \{\beta^n\} = \begin{cases} 1 & iff \quad n \quad is \quad even \\ \beta & iff \quad n \quad is \quad odd \end{cases}.$$

Now,

$D_{16} = N \bullet H = \langle\alpha, \beta\rangle = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7, \alpha^8, \alpha^9, \alpha^{10}, \alpha^{11}, \alpha^{12}, \alpha^{13}, \alpha^{14}, \alpha^{15}, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta, \alpha^6\beta, \alpha^7\beta, \alpha^8\beta, \alpha^9\beta, \alpha^{10}\beta, \alpha^{11}\beta, \alpha^{12}\beta, \alpha^{13}\beta, \alpha^{14}\beta, \alpha^{15}\beta\}$

where $\bullet$ is the product of permutations. Thus, the subgroup $N$ is complemented in $D_{16}$. Subsequently, $D_{16}$ split over $N$ and hence, $D_{16}$ is separable.

The center $Z(D_{16})$ of $D_{16}$ is the non-trivial subgroup $\{1, \alpha^8\}$ of $D_{16}$. It is clearly seen that $N \cap Z(D_{16}) \ne \{1\}$ and $H \cap Z(D_{16}) = \{1\}$. From our simple calculations, we observed that $r \ne 2^{n-1} + 1$. Hence, the split decompositions of $D_{16}$ are not isomorphic.*

It is also clear that theorem 2.1 above gives a presentation for those non-Abelian separable metacyclic 2-groups with $m = 1$ and $n \ge 3$, that have all split decompositions isomorphic. The following result follows directly from theorem 2.1 above. It presents other necessary and sufficient conditions for a metacyclic 2-group of the type mentioned above to have all its split decompositions isomorphic.

*THEOREM 2.3:* Supposed $G$ is a finite non-Abelian separable metacyclic 2-group of the type $G = \langle x, y \mid x^{2^n} = y^2 = 1, x^y = x^r\rangle$ with $n \ge 3$. Then the following statements are equivalent.

(i) All split decompositions of the group $G$ are of the form $G = [N]H$, where $N \lhd G$, $N$ is cyclic of order $2^n$ and $H$ is cyclic of order 2;

(ii) The order $|\Omega_1(G)| = 2^2$;

(iii) $Z(G) = \Phi(G)$;

(iv) $\Omega_{n-1}(G) \ne G$;

(v) The group $G$ is of class 2.

## III. SEPARABLE METACYCLIC 2-GROUPS; M ≥ 2

In this section, as shown by Kirtland [10] (see theorem 3.1) that if $G$ is a separable metacyclic 2-group with presentation $G = \langle x, y \mid x^{2^n} = y^{2^m} = 1, x^y = x^r\rangle$, $m \ge 2$, then $G$ must have all its split decompositions isomorphic. We therefore implement the result using GAP Software by constructing some finite groups whose subgroups are cyclic.

*THEOREM 3.1:* If $G$ is a separable metacyclic 2-group of type $G = \langle x, y \mid x^{2^n} = y^{2^m} = 1, x^y = x^r\rangle$ with $m \ge 2$, then all its split decompositions are isomorphic [10].

*PROOF.* If the group $G$ is Abelian, then the result follows directly from [11]. Now, supposed $G$ is not Abelian. Consequently, we shall have $r \ge 3$ and $n \ge 2$. We now consider three cases.

*CASE 1:* Let $a \in G$ where $a = x^\alpha y^{2^{m-1}}$ with $\alpha$ odd and $o(a) \ne 2$.

Now suppose $o(a) \ne 2$, then $a^2 = (x^\alpha y^{2^{m-1}})^2 = x^{\alpha + \alpha r^{2^{m-1}}} = 1$. This implies that $\alpha + \alpha r^{2^{m-1}} = 0 \pmod{2^n}$ or $\alpha(1 + r^{2^{m-1}}) = k2^n$ where $k$ is a positive integer. The fact that $\alpha$ is odd and $\alpha | k2^n$ implies that $\alpha | k$. Thus $1 + r^{2^{m-1}} = (k/\alpha)2^n$ or $r^{2^{m-1}} = (k/\alpha)2^n - 1$. Hence, $x^{y^{2^{m-1}}} = x^{r^{2^{m-1}}} = x^{(k/\alpha)2^n - 1} = x^{-1}$. Moreover, since $m \ge 2$, we have $x^{y^{2^{m-2}}} = x^s$ where $1 \le s \le 2^n - 1$ and $s$ is odd. This implies that $s^2 = -1 \pmod{2^n}$, a contradiction. Hence, $o(a) \ne 2$.

*CASE 2:* Let $\Omega_1(G) = \{1, x^{2^{n-1}}, y^{2^{m-1}}, x^{2^{n-1}} y^{2^{m-1}}\}$.

Now if $a = x^\alpha y^\beta$ is a nontrivial element of $G$ such that $o(a) \ne 2$ and $\alpha = 0$ or $\beta = 0$, then $a = y^{2^{m-1}}$ or $a = x^{2^{n-1}}$ respectively. If $\alpha \ne 0$ and $\beta \ne 0$, then $\beta = 2^{m-1}$ and $a = x^\alpha y^{2^{m-1}}$.

Suppose $n = 2$. Then by case 1, $\alpha = 2 = 2^{n-1}$. But if $n \ge 3$ and since $\langle y\rangle$ acts on $\langle x\rangle$, there is a homomorphism $\varphi: \langle y\rangle \to \mathrm{Aut}(\langle x\rangle)$. Suppose that $\mathrm{Ker}(\varphi) \ne \{1\}$. Then $y^{2^{m-1}} \in \ker(\varphi)$ and $x$ and $y^{2^{m-1}}$ commute. As a result, $1 = (x^\alpha y^{2^{m-1}})^2 = x^{2\alpha} y^{2^m} = x^{2\alpha}$. This implies that $\alpha = 2^{n-1}$ and $a = x^{2^{n-1}} y^{2^{m-1}}$.

Next, supposed $\mathrm{Ker}(\varphi) = \{1\}$. Denote $\mathrm{Aut}(\langle x\rangle)$ by $\{\varphi_1, \varphi_3, \ldots, \varphi_{2^n - 1}\}$, where $\varphi_i : x \to x^i$, for $i = 1, 3, \ldots, 2^n - 1$. By [13], we get $\mathrm{Aut}(\langle x\rangle) \cong L + K$, where $L$ is cyclic of order $2^{n-2}$ and generated by $\varphi_5$, and $K$ is cyclic of order 2 generated by $\varphi_{2^n - 1}$. Then the only nontrivial elements of $\mathrm{Aut}(\langle x\rangle)$ of order 2 are $\varphi_{2^{n-1}+1}, \varphi_{2^n - 1}$, and $\varphi_{2^{n-1}-1}$ and since $|y^{2^{m-1}}| = 2$, it follows that $\varphi(y^{2^{m-1}}) = \varphi_{2^{n-1}+1}, \varphi_{2^n - 1}$, or $\varphi_{2^{n-1}-1}$. Again if $m \ge 2$, then $\varphi(y^{2^{m-2}}) = \varphi_j$ where $\varphi(y^{2^{m-1}}) = (\varphi(y^{2^{m-2}}))^2 = (\varphi_j)^2$. Thus $j^2 = 2^{n-1} + 1 \pmod{2^n}$, $2^n - 1 \pmod{2^n}$, or $2^{n-1} - 1 \pmod{2^n}$. But since $j^2 \ne -1$

(mod $2^t$) for any integer $t > 0$, we have $j^2 = 2^{n-1}$ (mod $2^n$) and $x^{y^{2^{m-1}}} = x^{2^{n-1}+1}$.

Finally, given that $a^2 = 1$, then $(x^\alpha y^{2^{m-1}})^2 = x^{\alpha+\alpha(2^{n-1}+1)} = x^{\alpha(2^{n-1}+2)} = 1$ and since $|x^{2^{m-1}+2}| = 2^{n-1}$, it follows that $\alpha = 2^{n-1}$. Hence, $x^\alpha y^\beta = x^{2^{n-1}} y^{2^{m-1}}$ and $\Omega_1(G) = \{1, x^{2^{n-1}}, y^{2^{m-1}}, x^{2^{n-1}} y^{2^{m-1}}\}$.

*CASE 3:* All split decompositions of G are isomorphic.

Supposed that $G$ is given by the presentation $G(n.m.r) = \langle x, y \mid x^{2^n} = y^{2^m} = 1, x^y = x^r \rangle$, and that $G = [N]H$. Then $G(n.m.r)/\langle x^{2^{n-1}} \rangle \cong G(n-1, m, r)$ and if $n \geq 2$, then the result is true by induction on $n$.

Again, since $[x, y] = x^{-1}x^y = xr^{-1}$, it implies that $G' = \langle xr^{-1} \rangle \leq \langle x^2 \rangle$. Now let $z = x^{2^{n-1}} \in \Omega_1(G') \cap Z(G)$. If $z \notin N$, then $[G, N] \leq G' \cap N = \langle x^{r-1} \rangle \cap N = \{1\}$. Thus $N \leq Z(G)$ and so, $G = N \times H$. But this yields $|\Omega_1(G)| = 4$. Thus, we have $|\Omega_1(N)| = |\Omega_1(H)| = 2$. Consequently, $N$ and $H$ each have only one subgroup of order 2. But by [12], $N$ and $H$ are either cyclic or a generalized quaternion group. In either case, both $N$ and $H$ are inseparable and thus by [11], $N \cong \langle x \rangle$ and $H \cong \langle y \rangle$, or $N \cong \langle y \rangle$ and $H \cong \langle x \rangle$. In either case, $H$ is cyclic, implying that $G$ is abelian, a contradiction.

Hence, $\langle z \rangle \leq N$ and $[\langle x \rangle/\langle z \rangle]\langle y \rangle \cong G(n-1, m, r) \cong G/\langle z \rangle \cong [N/\langle z \rangle]H\langle z \rangle/\langle z \rangle$. As a result, $N/\langle z \rangle \cong \langle x \rangle/\langle z \rangle$ and $H \cong \langle y \rangle$ or $N/\langle z \rangle \cong \langle y \rangle$ and $H \cong \langle x \rangle/\langle z \rangle$. In either case, $H$ is cyclic and $N$ is abelian. Also if $N/\langle z \rangle \cong \langle x \rangle/\langle z \rangle$, then $N$ has order $2^n$ and is of exponent at least $2^{n-1}$. Thus $N$ is isomorphic to $Z_{2^n}$ or $Z_{2^{n-1}} \times Z_2$. Finally, if $N \cong Z_{2^{n-1}} \times Z_2$, then we have $|\Omega_1(N)| = 4$ and $|\Omega_1(G)| \geq 8$. This is a contradiction. Thus, $N$ is cyclic and if $N/\langle z \rangle \cong \langle y \rangle$, then a similar argument yields that $N$ must also be cyclic. Hence, both $N$ and $H$ are cyclic and by [1], $N \cong \langle x \rangle$ and $H = \langle y \rangle$.*

## A. IMPLEMENTATION OF THEOREM 3.1

Given a structured object $X$ of any sort, symmetry is a mapping of the object onto itself which preserves structure. But if $X$ is a finite set with no additional structure, then symmetry is defined as a bijective mapping from the set to itself, giving rise to what is called permutation group $S_n$. In this section we generate the non-Abelian group $S_{16}$ in GAP and then find some of its subgroups with one, two and three generators as follows.

```
gap> S:= SymmetricGroup(16);
Sym( [ 1 .. 16 ] )
gap> C:= CyclicGroup(IsPermGroup, 16);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16) ])
gap> r:= (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16);;
gap> s:= (2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10);;
gap> K:= Subgroup(S, [r,s]);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16), (2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10) ])
gap> Size(S);
20922789888000
gap> Factors(20922789888000);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 5, 5, 5, 7, 7, 11, 13 ]
gap> D:= DirectProduct(C, K);
<permutation group with 3 generators>
gap> Size(C); Size(K); Size(D);
16
32
512
gap> IsNormal(D, C);
true
gap> IsNormal(D, K);
true
gap> IsCyclic(K);
false
gap> Size(C)*Size(K) = Size(D);
true
gap> a:= (1,3)(4,16)(5,15)(6,14)(7,13)(8,12)(9,11);;
gap> R:= Subgroup(S, [r]);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16) ])
gap> Size(R);
16
gap> L:= Subgroup(S, [r,a]);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16), (1,3)(4,16)(5,15)(6,14)(7,13)(8,12)(9,11) ])
gap> Size(L);
32
gap> IsNormal(D, L);
true
gap> Size(R)*Size(L) = Size(D);
true
gap> IsAbelian(S);
false
gap> IsAbelian(D);
false
```

From the above results, the non-Abelian group $D \subseteq S_{16}$ is the semidirect product of the subgroups $C$ and $K$ of $S_{16}$ i.e. $D = C \times K$. But $D = R \times L$ with $C \cong R$, $K \cong L$. Also, $L \lhd D$ and $K \lhd D$. Thus, both $L$ and $K$ splits over $D$ and we write $D = [K]C = [L]R$. Hence, the two split decompositions $[K]C$ and $[L]R$ are isomorphic.

Next, we generate a cyclic group $G$ as a subgroup of $S_{16}$ as follows.

```
gap> G:= CyclicGroup(IsPermGroup, 16);
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16) ])
gap> Elements(G);
[ (), (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16), (1,3,5,7,9,11,13,15)(2,4,6,8,10,12,14,16),
  (1,4,7,10,13,16,3,6,9,12,15,2,5,8,11,14), (1,5,9,13)(2,6,10,14)(3,7,11,15)(4,8,12,16),
  (1,6,11,16,5,10,15,4,9,14,3,8,13,2,7,12), (1,7,13,3,9,15,5,11)(2,8,14,4,10,16,6,12),
  (1,8,15,6,13,4,11,2,9,16,7,14,5,12,3,10), (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16),
  (1,10,3,12,5,14,7,16,9,2,11,4,13,6,15,8), (1,11,5,15,9,3,13,7)(2,12,6,16,10,4,14,8),
  (1,12,7,2,13,8,3,14,9,4,15,10,5,16,11,6), (1,13,9,5)(2,14,10,6)(3,15,11,7)(4,16,12,8),
  (1,14,11,8,5,2,15,12,9,6,3,16,13,10,7,4), (1,15,13,11,9,7,5,3)(2,16,14,12,10,8,6,4),
  (1,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2) ]
gap> a:= G.1;
(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16)
gap> Size(G);
16
gap> f:= x -> x^6;
```

```
function( x ) ... end
gap> N:= Subgroup(G, [f(a)]);
Group([ (1,7,13,3,9,15,5,11)(2,8,14,4,10,16,6,12) ])
gap> Elements(N);
[ (), (1,3,5,7,9,11,13,15)(2,4,6,8,10,12,14,16),
(1,5,9,13)(2,6,10,14)(3,7,11,15)(4,8,12,16),
  (1,7,13,3,9,15,5,11)(2,8,14,4,10,16,6,12),
(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16),
  (1,11,5,15,9,3,13,7)(2,12,6,16,10,4,14,8),
(1,13,9,5)(2,14,10,6)(3,15,11,7)(4,16,12,8),
  (1,15,13,11,9,7,5,3)(2,16,14,12,10,8,6,4) ]
gap> Size(N);
8
gap> IsNormal(G, N);
true
gap> f:= x -> x^5;
function( x ) ... end
gap> K:= Subgroup(G, [f(a)]);
Group([ (1,6,11,16,5,10,15,4,9,14,3,8,13,2,7,12) ])
gap> Size(K);
16
gap> b:= (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16);;
gap> M:= Subgroup(G, [b]);
Group([ (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) ])
gap> Elements(M);
[ (), (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) ]
gap> Size(M);
2
gap> Size(G) = Size(M)*Size(N);
true
```

The subgroup $N$ of $G$ generated by the function $f(x) = x^6$ for an element $x \in G$ is a normal subgroup while the function $f(x) = x^5$ generated the group $K \cong G$. Hence, $N$ is complemented in $G$.

## IV. CONCLUSION

In this paper, we have successfully shown that if $G$ is a finite separable metacyclic 2-group with presentation $G = \langle x, y \mid x^{2^n} = y^{2^m} = 1, x^y = x^r \rangle$, $m \geq 1$, $n \geq 3$, then all split decompositions of $G$ are either isomorphic (if $G$ is Abelian), or $G$ is the Dihedral group of order $2n$. We therefore conclude that every finite group with the given presentation can be expressed as a semidirect product of a cyclic group with another cyclic group.

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