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Chromatic Complementary Tree Domination In Graphs

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***Abstract:** Let G = (V, E) be a simple graph. A subset D of V(G) is called a complementary tree dominating set (ctd-set) of G if D is a dominating set and <V–D> is a tree. D is called a chromatic complementary tree dominating set (chromatic ctd-set) of G if D is a ctd-set and χ(<D>) = χ(G). The minimum cardinality of a chromatic ctd-set of G is denoted by $\gamma^{\chi}_{ctd}(G)$ and is called chromatic complementary tree domination number of G. In this paper, a study of chromatic ctd-set is initiated.*

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***Keywords:** Complementary tree dominating set, chromatic ctd-set, chromatic ctd numbers.*

I. CHROMATIC COMPLEMENTARY TREE DOMINATING SET

DEFINITION 1:

Let G = (V, E) be a simple graph. A subset D of V(G) is called a complementary tree dominating set (ctd-set) of G if D is a dominating set and <V–D> is tree.

DEFINITION 2:

Let G = (V, E) be a simple graph. A subset D of V(G) is called a chromatic complementary tree dominating set (chromatic ctd-set) of G if D is a complementary tree dominating set of G and χ(<D>) = χ(G).

Since V(G) is a ctd-set of G and χ(<V(G)>) = χ(G), V(G) is a chromatic ctd-set of G.

The minimum cardinality of a chromatic ctd-set of G is called the chromatic complementary tree domination number and is denoted by $\gamma^{\chi}_{ctd}(G)$.

Any chromatic ctd-set is also a ctd-set and hence $\gamma_{ctd}(G) \le \gamma^{\chi}_{ctd}(G)$.

EXAMPLE 3:

If $G \cong C_4$, then $\gamma_{ctd}(G) = \gamma^{\chi}_{ctd}(G)$ and if $G \cong K_4 - e$, $\gamma_{ctd}(G) = 1$ and $\gamma^{\chi}_{ctd}(G) = 3$.

THEOREM 4:

Let G = P_n, then

$$\gamma^{\chi}_{ctd}(P_n) = \begin{cases} n-1 & \text{if } n = 4 \\ n-2 & \text{if } n \ge 5 \end{cases}$$

PROOF:

Let V(P_n) = {$u_1, u_2, \ldots, u_n$}

CASE (I): $n \ge 5$

The set D = {$u_1, u_2, \ldots, u_{i-1}, u_{i+2}, \ldots, u_{n-1}, u_n$} is a γ_{ctd}-set of P_n and V–D = {u_i, u_{i+1}}. Since, <D> $\cong P_{i-1} \cup P_{n-i-1}$, χ(<D>) = 2 = $\chi(P_n)$. Therefore, D is a chromatic ctd-set of G. Also, n–2 = $\gamma_{ctd}(G) \le \gamma^{\chi}_{ctd}(G) \le |D|$ = n–2. Hence, $\gamma^{\chi}_{ctd}(G)$ = n–2 if $n \ge 5$.

CASE (II): n = 4

Then the set D = {u_1, u_4} is a γ_{ctd}-set of G and χ(<D>) = 1, Since D is independent. But $\chi(P_4) = 2$.

Since, $\chi(D \cup \{u_2\}) = 2 = \chi(P_4)$, $D \cup \{u_2\}$ is a chromatic ctd-set of G.

Therefore, $\gamma^{\chi}_{ctd}(G) \le |D \cup \{u_2\}| \le 3$ = n–1. But, $\gamma^{\chi}_{ctd}(G) > \gamma_{ctd}(G) = |D| = 2$. That is, $\gamma^{\chi}_{ctd}(G) \ge 3$ = n–1. Hence, $\gamma^{\chi}_{ctd}(G)$ = n–1 if n = 4.

□

THEOREM 5:

Let G = C_n, $n \ge 4$

$$\gamma^{\chi}_{ctd}(C_n) = \begin{cases} n-2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

PROOF:

Let $V(C_n) = \{u_1, u_2, ..., u_n\}$. Then the set $D = \{u_1, u_2, ..., u_{n-2}\}$ is a γ_{ctd}-set of G and $<D> \cong P_{n-2}$ and hence $\chi(<D>) = 2$. But, $\chi(G) = \chi(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$

CASE (I): n is even

Since, $\chi(<D>) = \chi(C_n) = 2$, D is a chromatic ctd-set of G and hence, $n-2 = \gamma_{ctd}(G) \leq \gamma^{\chi}_{ctd}(G) \leq |D| = n-2$. Therefore, $\gamma^{\chi}_{ctd}(G) = n-2$.

CASE (II): n is odd.

Here, $\chi(<D>) = 2$ and $\chi(C_n) = 3$. Let $D' = D \cup \{u_{n-1}\}$. Then D′ is a ctd-set and $<D> \cong P_{n-1}$ and $\chi(<D'>) = 2 \neq \chi(C_n)$. Therefore, D′ is not a chromatic ctd-set of G.

Therefore, the set $D' \cup \{u_n\} = V(G)$ is a chromatic ctd-set of G and $|D' \cup \{u_n\}| = n$, if n is even.

Hence, $\gamma^{\chi}_{ctd}(C_n) = \begin{cases} n-2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$ □

THEOREM 6:

$\gamma^{\chi}_{ctd}(G)\ (K_{m,n}) = m$ if $m \leq n$ and $m, n \geq 2$.

PROOF:

Let $G = K_{m,n}$, $m, n \geq 2$ and $m \leq n$ and let $\{v_1, v_2\}$ be the bipartition of G. Let $V_1 = \{u_1, u_2, ..., u_m\}$, $V_2 = \{v_1, v_2, ..., v_n\}$. Then the set $D = \{v_1, u_1, u_2, ..., u_{m-1}\}$ is a ctd-set of $K_{m,n}$ and $<D> \cong K_{1,m-1}$. Therefore, $\chi(<D>) = \chi(G)$ and hence D is a chromatic ctd-set of G which implies $\gamma^{\chi}_{ctd}(G) \leq m$. Also $m = \gamma_{ctd}(G) \leq \gamma^{\chi}_{ctd}(G) \leq m$. □

THEOREM 7:

If $G \cong K_{1,n}$, then $\gamma^{\chi}_{ctd}(G) = n$, where $n \geq 2$.

PROOF:

Let $V(K_{1,n}) = \{u, u_1, u_2, ..., u_n\}$, where u is the central vertex and $u_1, u_2, ..., u_n$ are the pendant vertices of $K_{1,n}$.

The set $D = V(K_{1,n}) - \{u_n\}$ is a ctd-set of $K_{1,n}$ and $<D> \cong K_{1,n-1}$. Also, $\chi(<D>) = \chi(K_{1,n}) = 2$.

D is a minimum chromatic ctd-set of $K_{1,n}$.

Hence, $\gamma_{ctd}(K_{1,n}) = n$. □

Note. If $G \cong K_n$, then $\gamma^{\chi}_{ctd}(G) = n$, $n \geq 3$.

THEOREM 8:

For a tree T, $\gamma^{\chi}_{ctd}(T) = m+1$, if each vertex of degree atleast two is a support, where m is the number of pendant vertices in T.

PROOF:

Assume each vertex of degree atleast two of T is a support. Let D be the set of pendant vertices in T. Then D is a ctd-set of G and $\chi(<D>) = 1$. But $\chi(T) = 2$.

Therefore, $\gamma^{\chi}_{ctd}(G) > |D| = m$.

That is, $\gamma^{\chi}_{ctd}(G) \geq m+1$.

Let u be a vertex in T of degree atleast two and is adjacent to exactly one support of T.

Then $D \cup \{u\}$ is a ctd-set of G and $<D \cup \{u\}> \cong <T-\{u\}>$ and $\chi(<D \cup \{u\}>) = \chi<T-\{u\}> = 2$. Therefore, $D \cup \{u\}$ is a chromatic ctd-set of G.

$\gamma^{\chi}_{ctd}(T) \leq |D \cup \{u\}| = m+1$

Hence, $\gamma^{\chi}_{ctd}(T) = m+1$. □

THEOREM 9:

Let T be a tree, which is not a star. Then $\gamma^{\chi}_{ctd}(T) \leq \gamma_{ctd}(T) + 1$.

PROOF:

Let T be a tree on p vertices where $p \geq 4$.

Let D be a ctd-set of T, since T is not a star, $|D| \leq p-2$ and $\chi(T) = 2$.

CASE (I): D is independent

Then all the pendant vertices of T are numbers of D. That is, every vertex in V–D is a support of G. $\chi(D) = 1$ and $\chi(T) = 2$. Let v be a vertex of minimum degree in T, such that v is a pendant vertex in $<V(T)-D>$.

Therefore, $\chi(<D \cup \{u\}>) = 2$ and $\gamma^{\chi}_{ctd}(T) = |D \cup \{v\}| = \gamma_{ctd}(G) + 1$

CASE (II): D is not independent Then $\chi(<D>) = 2 = \chi(T)$. Therefore, D is a chromatic ctd-set of T.

Therefore, $\gamma^{\chi}_{ctd}(T) = \gamma_{ctd}(T) < \gamma_{ctd}(T) + 1$. □

The above inequality holds, if every vertex of degree atleast 2 in T is a support.

THEOREM 10:

If $G \cong T + K_1$, where T is a tree, then $\gamma^{\chi}_{ctd}(G) \leq t+2$ where t is the minimum number of pendant vertices adjacent to an end support vertex of T.

PROOF:

Let $G \cong T + K_1$, where T is a tree. Then, $\gamma_{ctd}(G) = 1$. Let D be a γ_{ctd}-set of G such that $|D| = 1$. But $\chi(T + K_1) = 3$

Therefore, D is not a chromatic ctd-set of G. Let $V(K_1) = \{u\}$ and let v be a support of T having minimum number of pendant vertices such that $T-\{v\}$ is connected.

If S is the set of pendant vertices in T adjacent to v, then $D \cup S \cup \{v\}$ is a ctd-set of G and $\chi(D \cup S \cup \{v\}) = 3$. $D \cup S \cup \{v\}$ is a chromatic ctd-set of G and hence $\gamma^{\chi}_{ctd}(G) \leq |S| + 2 = t + 2$. □

REMARK:

- ✓ $\gamma^{\chi}_{ctd}(G) = 1$ if and only if $G \cong K_2$.
- ✓ If G is a path, on atleast 5 vertices, then any γ_{ctd}-set of G is a chromatic ctd-set.

  That is, $\gamma^{\chi}_{ctd}(G) = \gamma_{ctd}(G)$.
- ✓ If G is a star on atleast p ($p \geq 3$) vertices, then $\gamma_{ctd}(G) = p-1$.

If D is a ctd-set of G containing the central vertex of G, then $\gamma^{\chi}_{ctd}(G) = \gamma_{ctd}(G)$.

- ✓ If G is a graph on atleast three vertices, then $\gamma^{\chi}_{ctd}(G) \geq 2$.

THEOREM 11:

For any connected graph, $\gamma^{\chi}_{ctd}(G) = 2$ if and only if G is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of K_2 such that $deg_G\ v \geq 2$, for all $v \in V(K_2)$ and no two adjacent vertices of the tree are adjacent to the same vertex of K_2.

PROOF:

Let D be a γ_{ctd}-set of G. Then $\gamma_{ctd}(G) = |D|$

But $\gamma^{\chi}_{ctd}(G) = 2$

Since $\gamma_{ctd}(G) \leq \gamma^{\chi}_{ctd}(G) = 2$, $|D| \leq 2$

Therefore, $|D| = 1$ or $|D| = 2$.

CASE (I): $|D| = 1$

Therefore, $G \cong T + K_1$, for any tree T on atleast 2 vertices and $\chi(G) \geq 3$, $\chi(<D>) = 1$ implies that, D is not a chromatic ctd-set of G.

CASE (II): $|D| = 2$

Then G is one of the following graphs.

- ✓ G is the graph obtained from $T + K_1$ with one pendant edge attached at the vertex of K_1, where T is any tree.
- ✓ G is the graph obtained from a tree by joining each of the vertices of the tree to the vertices of K, such that $deg_G\ v \geq 2$, $\forall\ v \in V(K_2)$.
- ✓ G is the graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2K_1$ such that $deg_G\ v \geq 1$, $\forall\ v \in V(2K_1)$.

If G is one of the graphs mentioned in (i) then $\gamma_{ctd}(G) \geq 3$ and in (iii), $\chi(<D>) = 1$.

In (ii), if any two adjacent vertices of the tree are adjacent to the same vertex of K_2, then $\gamma_{ctd}(G) \geq 3$.

Therefore, G is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of K_2 such that $deg_G\ v \geq 2$, for all $v \in V(K_2)$ and no two adjacent vertices of the tree are adjacent to the same vertex of K_2.

In this case, $D = V(K_2)$ is a γ_{ctd}-set of G and $\chi(<D>) = 2$. Also, $\chi(G) = 2$.

Therefore, D is a chromatic ctd-set of G.

THEOREM 12:

Given a positive integer $t \geq 3$, there exists a connected graph G such that $\gamma^{\chi}_{ctd}(G) = t$.

PROOF:

Let G be a graph obtained by attaching a vertex of complete graph on t vertices at the vertex of K_1 in $P_n + K_1$, where P_n is a path on n vertices $(n \geq 2)$.

For this graph G, $\chi(G) = t$

The set D of vertices in K_t is a ctd-set of G and $\chi(D) = t$.

Therefore, D is a chromatic ctd-set of G.

THEOREM 13:

$$\gamma^{\chi}_{ctd}(W_n) = \begin{cases} n & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

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