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New Generalizations Of Exponentail Distribution

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Abstract: ***The main purpose of this paper is to present k-Generalized Exponential Distribution which among other things include Generalized Exponential and Weibull Distributions as special cases. Besides, we also obtain three parameters extension of Generalized Exponential Distribution. We shall also discuss moment generating functions (m.g.f's) of these newly introduced distributions.***

Keywords: ***Generalized Exponential Distribution (GED), k-beta function, k-gamma function, k-Generalized Exponential Distribution (k-GED) and Weibull distribution.***

I. INTRODUCTION

The gamma family distributions was discussed by Karl Pearson in 1895. As pointed out in Balakrishnan and Basu (1995). However, after a period of 35 years the exponential distribution, which is a special case of the gamma distribution to appear on its own. It is also related to Poisson process as it has been observed that the time between two successive Poisson events follows the exponential distribution. While discussing the sampling of standard deviation (S.D), the exponential distribution was referred by Kondo (1930) as Pearson's Type X distribution. Steffensens (1930), Teissier (1934) and Weibull (1939) proposed the applications of exponential distribution in actuarial, biological and engineering problems respectively.

An extension of exponential distribution was proposed by Weibull (1951). The exponential distribution is a special case wherein the shape parameter equals one. The Weibull distribution has many applications in survival analysis and reliability engineering, for reference see Lai et.al (2006). Some other applications in industrial quality control are discussed in Berrettoni (1964).

II. SOME BASIC DEFINITIONS AND K-GENERALIZED EXPONENTIAL DISTRIBUTIONS

We begin with some definitions which provide a base for the definition of K-Generalized Exponential Distributions.

✓ The Euler gamma function $\Gamma(\alpha)$ is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt\ , \quad \mathrm{R}(\alpha) > 0.$$

For a random variable X is said to have gamma distribution with parameter $\alpha > 0$, if its p.d.f. is given by

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}, \qquad 0 < x < \infty$$
$$= 0, \quad \text{elsewhere.}$$

Replacing x by $\frac{x}{\lambda}$, we get the following form of gamma distribution with parameters α, λ with $\alpha > 0$ and $\lambda > 0$,

$$f(x) = \frac{1}{\lambda^\alpha\, \Gamma(\alpha)}\ x^{\alpha-1} e^{-\frac{x}{\lambda}}, \qquad 0 < x < \infty,\ \alpha > 0,\ \lambda > 0.$$

The Gamma distribution with parameters α, λ often arises in practices, as the distribution of time one has to wait until a fixed number of events have occurred.

- ✓ The beta function of two variables m and n is defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \mathrm{Re}(m)>0, \mathrm{Re}(n)>0. \quad (1)$$

If we take $x = 1-y$,

then, $0 \le x \le 1$ implies $0 \le y \le 1$, and we get

$$B(m,n) = \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1} dx$$

$$= B(n,m) \quad (2)$$

In the literature (for reference see [1-3]), it is known that

$$B(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (3)$$

In (2), if we take

$$x = \frac{1}{z+1}$$

then $0 \le x \le 1$ implies $0 \le z < \infty$,and we get

$$B(n,m) = \int_\infty^0 \left(\frac{1}{z+1}\right)^{n-1}\left(\frac{z}{z+1}\right)^{m-1}\left(-\frac{1}{(z+1)^2}\right)dz$$

$$= \int_0^\infty \frac{z^{m-1}}{(z+1)^{m+n}} dz.$$

- ✓ A continuous random variable X is said to have beta distribution with parameter m and n, if its p.d.f. is given by

$$f(x) = \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}, \quad 0 \le x \le 1$$

$= 0$, elsewhere.

This distribution is known as beta distribution of 1st kind (for reference see [2]).

The beta distribution has an application to model a random phenomenon whose set of possible values is a finite interval [a, b], which by letting a denote the origin and taking $(b-a)$ as a unit measurement can be transformed into the interval [0, 1].

- ✓ A continuous random variable X is said to have beta distribution of 2nd kind with parameter m and n, if its p.d.f. is given by

$$f(x) = \frac{1}{B(m,n)} \frac{z^{m-1}}{(1+z)^{m+n}}, \quad 0 \le z < \infty, \qquad m, n>0.$$

$= 0$, elsewhere.

- ✓ More recently, G.Rahman, S.Mubeenet.al.,[2] (for more details see [3-11]) have defined k-gamma and k-beta distributions and their m.g.f's as follows:

For $k > 0$ and $z \in \mathbb{C}$, the k-gamma function is defined by the integral

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt.$$

For $\mathrm{R}(m) > 0$, $\mathrm{R}(n) > 0$, the k-beta function of two variables m and n is defined by

$$B_k(m,n) = \frac{1}{k}\int_0^1 t^{\frac{m}{k}-1}(1-t)^{\frac{n}{k}-1} dt.$$

It is implicit in the literature (for reference see [2]),

$$B_k(m,n) = \frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)}. \quad (4)$$

For the sake of completeness, we present a very simple proof of the relation (4).

We have

$$B_k(m,n) = \frac{1}{k}\int_0^1 x^{\frac{m}{k}-1}(1-x)^{\frac{n}{k}-1} dx.$$

Put $x = \cos^2\theta$, then

$dx = -2\cos\theta\sin\theta\, d\theta$, and we get

$$B_k(m,n) = \frac{2}{k}\int_0^{\pi/2} (\cos\theta)^{\frac{2m}{k}-1} (\sin\theta)^{\frac{2n}{k}-1} d\theta. \quad (5)$$

Now

$$\Gamma_k(m) = \int_0^\infty t^{m-1} e^{-\frac{t^k}{k}} dt.$$

Put $\frac{t^k}{k} = x^2$ or $t = (kx^2)^{1/k}$, so that

$$dt = \frac{2}{k} k^{1/k} x^{\frac{2}{k}-1} dx$$

$$= 2k^{\frac{1}{k}-1} x^{\frac{2}{k}-1} dx,$$

we get

$$\Gamma_k(m) = 2k^{\frac{1}{k}-1}\int_0^\infty k^{\frac{m-1}{k}} x^{\frac{2}{k}(m-1)} e^{-x^2} x^{\frac{2}{k}-1} dx$$

$$= 2k^{\frac{m}{k}-1}\int_0^\infty x^{\frac{2m}{k}-1} e^{-x^2} dx. \qquad (6)$$

Since the integrals involved are convergent, we have

$$\Gamma_k(m)\,\Gamma_k(n) =$$

$$\left(2k^{\frac{m}{k}-1}\int_0^\infty x^{\frac{2m}{k}-1} e^{-x^2} dx\right)\left(2k^{\frac{n}{k}-1}\int_0^\infty y^{\frac{2n}{k}-1} e^{-y^2} dy\right),$$

$$= 4k^{\frac{m+n}{k}-2}\int_0^\infty\int_0^\infty x^{\frac{2m}{k}-1} y^{\frac{2n}{k}-1} e^{-(x^2+y^2)} dx\,dy.$$

Put $x = r\cos\theta,\ y = r\sin\theta$,

so that

$$dx\,dy = r\,dr\,d\theta,$$

we get

$$\Gamma_k(m)\,\Gamma_k(n) = 4k^{\frac{m+n}{k}-2}\int_0^\infty\int_0^{\pi/2} (\cos\theta)^{\frac{2m}{k}-1}(\sin\theta)^{\frac{2n}{k}-1} e^{-r^2} r^{\frac{2m}{k}+\frac{2n}{k}-1} dr\,d\theta$$

$$= \left\{\frac{2}{k}\int_0^{\pi/2}(\cos\theta)^{\frac{2m}{k}-1}(\sin\theta)^{\frac{2n}{k}-1} d\theta\right\}\left\{2k^{\frac{m+n}{k}-1}\int_0^\infty r^{\frac{2(m+n)}{k}-1} e^{-r^2} dr\right\}.$$

Using (5) and (6), we get

$$\Gamma_k(m)\,\Gamma_k(n) = B_k(m,n)\,\Gamma_k(m+n),$$

which gives

$$B_k(m,n) = \frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)}.$$

This completes the proof of the relation (4).

COROLLARY 1: We have

- ✓ $\Gamma_k(k) = 1.$
- ✓ $B_k\left(\frac{k}{2},\frac{k}{2}\right) = \frac{\pi}{k}.$
- ✓ $\Gamma_k\left(\frac{k}{2}\right) = \sqrt{\frac{\pi}{k}}.$

PROOF: Put $m = k$ in (6), we get

$$\Gamma_k(k) = 2\int_0^\infty x e^{-x^2} dx$$

$$= -\left[e^{-x^2}\right]_0^\infty = 1.$$

From (4), we have

$$B_k(m,n) = \frac{\Gamma_k(m)\Gamma_k(n)}{\Gamma_k(m+n)}.$$

We take $m = n = \frac{k}{2}$, we get

$$B_k(k/2,k/2) = \frac{\Gamma_k(k/2)\Gamma_k(k/2)}{\Gamma_k(k)}$$

$$= \Gamma_k(k/2)\Gamma_k(k/2)$$

$$= [\Gamma_k(k/2)]^2.$$

This gives

$$[\Gamma_k(k/2)] = \sqrt{B_k(k/2,k/2)}. \qquad (7)$$

Putting $m = \frac{k}{2}$ and $n = \frac{k}{2}$ in (5), we get

$$B_k(k/2,k/2) = \frac{2}{k}\int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{2}{k}[\theta]_0^{\pi/2}$$

$$= \left(\frac{2}{k}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{k},$$

which establishes (ii).

Putting the value of $B_k(k/2,k/2)$ in (7), we get

$$\Gamma_k\left(\frac{k}{2}\right) = \sqrt{\frac{\pi}{k}},$$

which proves (iii).

A random variable X of continuous type is said to have Weibull distribution if its probability density function is given by

$$f(x,\alpha,\beta,k) = \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}, \quad x > \nu$$

$$= 0, \quad \text{elsewhere}, \quad x \le \nu.$$

Weibull distribution is widely used in engineering practice due to its versatility. It was originally proposed for the interpretation of fatigue data but now it is also used for many other problems in engineering. In particular in the field of life phenomenon, it is used as the distribution of life time of some object, particularly when the "weakest link" model is appropriate for the object, that is, consider an object consisting of many parts and suppose that the object experiences death (failure) when any of its parts fail. It has been shown [1] (both the oretically and empirically) under these conditions that Weibull distribution provides a close approximation to the distribution of the life time of the item.

The Gamma and Weibull distributions are commonly used for analyzing any life time data or skewed data. Both distributions have nice physical interpretation and several desirable properties. Unfortunately both distributions have drawbacks, one major disadvantage of the gamma distribution is that the distribution function or the survival function cannot be computed easily if the shape parameter is not an integer. By using mathematical tables or computer software one obtains the distribution function, the survival function or hazard function. This makes the gamma distribution unpopular as compared to Weibull distribution whose distribution function, hazard function or survival function is easy to compute. It is well known that even though the Weibull distribution has convenient representation of distribution function, the distribution of the sum of independent and identically

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distributed (i.i.d) Weibull random variables is not simple to obtain. Therefore, the distribution of the mean of random sample from Weibull distribution is not easy to compute whereas the distribution of sum of independent and identically distributed (i.i.d) gamma random variables is well known. For more details see Mudholkar, Srivastava and Freimer (1995), Mudholkar and Srivastava (1993), Gupta and Kundu (1997) and Gupta R.C (1998).

Recently R.D. Gupta and D.Kundu have introduced three parameter Exponential Distribution (location, scale, shape) and study the theoretical properties of this family and compared them with respective well studies properties of Gamma and Webull distributions. The increasing and decreasing hazard rate of the Generalized Exponential Distribution (GED) depends on the shape parameter. Generalized Exponential Distribution (GED) has several properties that are quite similar to gamma distribution but it has distribution function similar to that of the Weibull distribution which can be computed easily. Since the Generalized Exponential family has the likelihood ratio ordering on the shape parameter, one can construct a uniformly most powerful test for testing one sided hypothesis on the shape parameter when the scale and location parameters are known.

III. A continuous random variable X whose probability density function (p.d.f.) is given by

$$f(x) = \begin{cases} \beta e^{-\beta x} & \text{if } \; x \geq 0 \\ 0 & \text{if } \; x < 0, \quad \beta > 0, \end{cases}$$

is said to have an exponential distribution.

The Generalized Exponential Distribution (GED) introduced by R.D.Gupta and D.Kundu [10] has p.d.f.

$$f(x,\alpha,\beta) = \alpha\beta\left(1-e^{-\beta x}\right)^{\alpha-1} e^{-\beta x}, x>0 \text{ for } \alpha,\beta>0 \quad (8)$$

=0, otherwise.

The m.g.f. of (8) is given by

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx}\alpha\beta\left(1-e^{-\beta x}\right)^{\alpha-1} e^{-\beta x}dx$$

$$= \alpha \mathrm{B}\left(\alpha, 1-(t/\beta)\right), \quad (9)$$

where $\mathrm{B}(m,n) = \int_0^1 y^{n-1}(1-y)^{m-1}dy.$

Replacing β by $\frac{1}{\lambda}$ and x by $(x-\mu)$ in (8), we get the following form of Generalized Exponential Distribution (GED).

$$f(x,\alpha,\lambda,\mu) = \frac{\alpha}{\lambda}\left(1-e^{-\frac{(x-\mu)}{\lambda}}\right)^{\alpha-1} e^{-\frac{(x-\mu)}{\lambda}}, \quad (10)$$

$x > \mu,\ \alpha > 0,\ \beta > 0,\ \lambda > 0$.

- ✓ The main aim of this paper is to present interesting extensions of Generalized Exponential Distribution (GED) in various ways and to study their moment generating functions (m.g.f's).We shall first define Generalized Exponential Distribution (GED) in terms of a new parameter k>0 and call it k-Generalized Exponential Distribution (k-GED). In fact, we prove the following result, which included Generalized Exponential Distribution asa special case.

THEOREM 1: Let X be a random variable of continuous type and let $\alpha > 0,\ \beta > 0,$ and $k > 0$ be the parameters, then the function

$$f(x,\alpha,\beta,k) = \alpha\beta\left(1-e^{-\beta\frac{x^k}{k}}\right)^{\alpha-1} x^{k-1} e^{-\beta\frac{x^k}{k}}, \; x>0$$

= 0, elsewhere

is the probability density function of random variable X of continuous type.

REMARK (1.1): If we take $k=1$, it reduces to Generalized Exponential Distribution.

PROOF OF THEOREM 1: Clearly

$f(x,\alpha,\beta,k) \geq 0$ for all $x > 0, \alpha > 0, \beta > 0, k > 0$.

Now

$$\int_0^\infty f(x,\alpha,\beta,k)dx = \int_0^\infty \alpha\beta\left(1-e^{-\beta\frac{x^k}{k}}\right)^{\alpha-1} x^{k-1} e^{-\beta\frac{x^k}{k}}dx$$

$$= \alpha\int_0^\infty \left(1-e^{-\beta\frac{x^k}{k}}\right)^{\alpha-1}\left(\beta x^{k-1} e^{-\beta\frac{x^k}{k}}\right)dx,$$

$$= \alpha\left[\frac{\left(1-e^{-\beta\frac{x^k}{k}}\right)^{\alpha}}{\alpha}\right]_0^\infty,$$

$$= \left[\left(1-e^{-\beta\frac{x^k}{k}}\right)^{\alpha}\right]_0^\infty,$$

$= 1 - 0 = 1.$

Hence $f(x,\alpha,\beta,k)$ is a p.d.f. of random variable X of continuous type.

IV. THE MOMENT GENERATING FUNCTION (M.G.F) OF THEOREM 1

In this section, we derive m.g.f. of the random variable X having k−Generalized Exponential Distribution (k-GED) in terms of new parameter k>0, we have

$$M_k(t) = E\left(e^{tX^k}\right)$$

$$= \int_0^\infty e^{tx^k} f(x,\alpha,\beta,k)\,dx$$

$$= \alpha\beta\int_0^\infty e^{tx^k}\left(1-e^{-\beta\frac{x^k}{k}}\right)^{\alpha-1}\left(x^{k-1}e^{-\beta\frac{x^k}{k}}\right)dx.$$

Put $e^{-\beta\frac{x^k}{k}} = y$,

then

$$e^{-\beta\frac{x^k}{k}}\left(-\beta x^{k-1}\right)dx = dy,$$

and

$$e^{x^k} = y^{-\frac{k}{\beta}}$$

so that

$$e^{tx^k} = y^{-\frac{tk}{\beta}}.$$

Therefore,

$$M_k(t) = \alpha\int_0^1 y^{-\frac{tk}{\beta}}(1-y)^{\alpha-1}dy$$

$$= \alpha\int_0^1 y^{(1-(tk/\beta))-1}(1-y)^{\alpha-1}dy$$

$$= \alpha\int_0^1 (1-y)^{\alpha-1}y^{(1-(tk/\beta))-1}dy$$

$$= \alpha\,\mathrm{B}\left(\alpha, 1-(tk/\beta)\right)$$

$$= \frac{\alpha\,\Gamma(\alpha)\,\Gamma\left(1-(tk/\beta)\right)}{\Gamma\left(\alpha+1-(tk/\beta)\right)}$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma\left(1-(tk/\beta)\right)}{\Gamma\left(\alpha+1-(tk/\beta)\right)}.$$

✓ Our next theorem is also a generalization of Exponential distribution in terms of new variable k, which includes Weibull distribution as a special case.

THEOREM 2: Let X be a random variable of continuous type and let $\alpha > 0,\ \beta > 0,$ and $k > 0$ be the parameters, then the function

$$f(x,\alpha,\beta,k) = k\alpha\beta\left(1-e^{-\beta x^k}\right)^{\alpha-1}x^{k-1}e^{-\beta x^k}, 0 < x < \infty$$

= 0, elsewhere,

is the probability density function (p.d.f.) of random variable X of continuous type.

REMARK (2.1): For $k=1$, K-Generalized Exponential Theorem 2 reduces to Classical Exponential Distribution.

PROOF OF THEOREM 2: Clearly

$f(x,\alpha,\beta,k) \geq 0$ for all $x > 0, \alpha > 0, \beta > 0,\ k > 0$.

Now

$$\int_0^\infty f(x,\alpha,\beta,k)\,dx = k\alpha\beta\int_0^\infty\left(1-e^{-\beta x^k}\right)^{\alpha-1}x^{k-1}e^{-\beta x^k}dx$$

$$= \alpha\int_0^\infty\left(1-e^{-\beta x^k}\right)^{\alpha-1}\left(k\beta x^{k-1}\right)e^{-\beta x^k}dx$$

$$= \alpha\left[\frac{\left(1-e^{-\beta x^k}\right)^\alpha}{\alpha}\right]_0^\infty$$

$$= \left[\left(1-e^{-\beta x^k}\right)^\alpha\right]_0^\infty$$

= 1- 0 = 1.

Hence $f(x,\alpha,\beta,k)$ is a p.d.f. of random variable X of continuous type.

REMARK (2.2): Weibull distribution is a special case of Theorem 2.

To see this, we take $\alpha=1,\ \beta=1$ in Theorem 2, it follows that

$$f(x,k) = k\,x^{k-1}e^{-x^k}$$

$$= e^{-x^k}\frac{d}{dx}\left(x^k\right), \text{ x>0}$$

$$= 0, \qquad x \leq 0$$

Replacing x by $\dfrac{x-\mu}{\alpha}$, we get

$$f(x,\alpha,k,\nu) = e^{-\left(\frac{x-\nu}{\alpha}\right)^k}\frac{d}{dx}\left(\frac{x-\nu}{\alpha}\right)^k, \qquad \left(\frac{x-\nu}{\alpha}\right) > 0$$

$$= 0, \qquad \left(\frac{x-\nu}{\alpha}\right) \leq 0$$

or

$$f(x,\alpha,k,\nu) = \frac{k}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{k-1}e^{-\left(\frac{x-\nu}{\alpha}\right)^k}, \qquad x > \nu,$$

$$= 0, \qquad x \leq \nu,$$

is a p.d.f. of X, which is clearly the density of Weibul Distribution.

V. THE MOMENT GENERATING FUNCTION (M.G.F) OF THEOREM 2

In this section, we derive m.g.f. of the random variable X having k−Generalized Exponential Distribution (k-GED) in terms of a new parameter k>0, we have

$$M_k(t) = E\left(e^{tX^k}\right)$$

$$= \int_o^\infty e^{tx^k} f(x,\alpha,\beta,k)\,dx$$

$$= k\alpha\beta\int_0^\infty e^{tx^k}\left(1-e^{-\beta x^k}\right)^{\alpha-1} x^{k-1}e^{-\beta x^k}dx.$$

Put $e^{-\beta x^k} = y$,

then

$$e^{x^k} = y^{-\frac{1}{\beta}},$$

and

$$e^{-\beta x^k}\left(-k\beta x^{k-1}\right)dx = dy.$$

Therefore,

$$M_k(t) = \alpha\int_0^1 y^{-\frac{t}{\beta}}(1-y)^{\alpha-1}dy$$

$$= \alpha\int_0^1 y^{(1-t/\beta)-1}(1-y)^{\alpha-1}dy$$

$$= \alpha\int_0^1 (1-y)^{\alpha-1}y^{(1-t/\beta)-1}dy$$

$$= \alpha\,\mathrm{B}\left(\alpha, 1-(t/\beta)\right)$$

$$= \frac{\alpha\Gamma(\alpha)\Gamma\left(1-(t/\beta)\right)}{\Gamma\left(\alpha+1-(t/\beta)\right)}$$

$$= \frac{\Gamma(\alpha+1)\Gamma\left(1-(t/\beta)\right)}{\Gamma\left(\alpha+1-(t/\beta)\right)}.$$

Next we present the following three parameters extension of Generalized Exponential Distribution (GED). In fact, we prove

THEOREM 3: Let X be a continuous random variable, then the function

$$f(x,\alpha,\beta,\delta) = \frac{\alpha\beta\delta}{1-(1-\delta)^\alpha}\left(1-\delta e^{-\beta x}\right)^{\alpha-1}e^{-\beta x}, \quad (11)$$

$$x>0, \alpha>0,\ \beta>0, \qquad 0<\delta\le 1$$

$= 0$, otherwise

is the p.d.f. of random variable X of continuous type.

PROOF OF THEOREM 3: Clearly $f(x,\alpha,\beta,\delta) \ge 0$ for all $x>0,\ \alpha>0,\ \beta>0,\ \delta>0$

Now

$$\int_0^\infty f(x,\alpha,\beta,\delta)\,dx = \frac{\alpha\beta\delta}{1-(1-\delta)^\alpha}\int_0^\infty\left(1-\delta e^{-\beta x}\right)^{\alpha-1}e^{-\beta x}dx$$

$$= \frac{\alpha}{1-(1-\delta)^\alpha}\int_0^\infty\left(1-\delta e^{-\beta x}\right)^{\alpha-1}\left(\beta\delta e^{-\beta x}\right)dx$$

$$= \left(\frac{\alpha}{1-(1-\delta)^\alpha}\right)\left[\frac{\left(1-\delta e^{-\beta x}\right)^\alpha}{\alpha}\right]_0^\infty$$

$$= \frac{\left(1-(1-\delta)^\alpha\right)}{\left(1-(1-\delta)^\alpha\right)} = 1.$$

This shows that $f(x)$ is a p.d.f. of the random variable X . This proves theTheorem 3.

If we replace β by $\frac{1}{\lambda}$ and x by $(x-\mu)$ in Theorem 3, we get the following form of Generalized Exponential Distribution (GED).

$$f(x,\alpha,\lambda,\mu,\delta) = \frac{\alpha\delta}{\lambda\left(1-(1-\delta)^\alpha\right)}\left(1-\delta e^{-\frac{(x-\mu)}{\lambda}}\right)^{\alpha-1}e^{-\frac{(x-\mu)}{\lambda}}, \quad (12)$$

$$x>\mu, \alpha>0,\ \lambda>0,\ \mu>0,\ 0<\delta\le 1.$$

$= 0$, otherwise.

REMARK (3.1): For $\delta=1$, Theorem 3 reduced to Generalized Exponential Distribution (GED).

REMARK (3.2): Taking $\delta=1$ in (12), we get relation (10).

VI. THE MOMENT GENERATING FUNCTION (M.G.F) OF THEOREM 3

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} f(x,\alpha,\beta,\delta)dx$$

$$= \frac{\alpha\beta\delta}{\left(1-(1-\delta)^\alpha\right)}\int_0^\infty e^{tx}\left(1-\delta e^{-\beta x}\right)^{\alpha-1}e^{-\beta x}dx$$

$$= \frac{\alpha\delta}{\left(1-(1-\delta)^\alpha\right)}\int_0^\infty\left(1-\delta e^{-\beta x}\right)^{\alpha-1}e^{tx}\left(\beta e^{-\beta x}\right)dx$$

Put $e^{-\beta x} = y$,

so that

$-\beta e^{-\beta x} dx = dy$ and $e^{tx} = y^{-\frac{t}{\beta}}$, we get

$$M(t) = \frac{\alpha\delta}{\left(1-(1-\delta)^{\alpha}\right)} \int_0^1 (1-\delta y)^{\alpha-1} y^{-\frac{t}{\beta}} dy, \qquad 0<\delta\le 1.$$

$$= \frac{\alpha}{\left(1-(1-\delta)^{\alpha}\right)} \mathrm{B}_{\delta}(\alpha, 1-t/\beta), \tag{13}$$

where $\mathrm{B}_{\delta} =$

$$\mathrm{B}_{\delta}(m,n) = \int_0^1 (1-\delta y)^{m-1} y^{n-1} dy . m>0, n>0, \ 0<\delta\le 1.$$

and(13) reduces to remark (3.1).

REMARK (3.3): For $\delta = 1$, we have

$$\mathrm{B}_1(m,n) = \int_0^1 (1-y)^{m-1} y^{n-1} dy = \mathrm{B}(m,n) .$$

REMARK (3.4): If in $\mathrm{B}_{\delta}(m,n) = \int_0^1 (1-\delta y)^{m-1} y^{n-1} dy$,

we put $x = \delta y$,

then

$dy = \frac{1}{\delta} dx$ and we get

$$\mathrm{B}_{\delta}(m,n) = \int_0^{\delta} (1-x)^{m-1} \left(\frac{x}{\delta}\right)^{n-1} \frac{1}{\delta} dx$$

$$= \frac{1}{\delta^n} \int_0^{\delta} (1-x)^{m-1} x^{n-1} dx, \qquad 0\le\delta\le 1.$$

REMARK (3.5): Letting $\delta \to 0$ in (11) and noting that

$$\underset{\delta\to 0}{Lt} . \frac{\alpha\delta}{\left(1-(1-\delta)^{\alpha}\right)} = 1, \text{ we get } f(x,\beta) = \beta e^{-\beta x},$$

if $x \ge 0, \ \beta > 0$,

which is the p.d.f. of the exponential distribution.

✓ Finally we present the following more general interesting result which among other things includes Weibull distribution as a limiting case.

THEOREM 4: Let X be a random variable of continuous type. If $\delta > 0, \ \beta > 0,$ and $k > 0$ are the parameters, then the function

$$f(x,\delta,\beta,k) = \frac{k\delta\beta}{1-(1-\delta)^k} \left(1-\delta e^{-x^{\beta}}\right)^{k-1} x^{\beta-1} e^{-x^{\beta}}, \ x>0, \ 0<\delta<1,$$

$= 0$, elsewhere, $x \le 0$,

is the p.d.f. of the random variable X.

PROOF OF THEOREM 4: Clearly

$f(x,\delta,\beta,k) \ \ge \ 0$ for all $x>0, \delta>0, \beta>0, \ k>0$.

Now

$$\int_0^{\infty} f(x,\delta,\beta,k)dx = k\delta\beta \int_0^{\infty} \left(1-\delta e^{-x^{\beta}}\right)^{k-1} x^{\beta-1} e^{-x^{\beta}} dx,$$

$$= \frac{k}{1-(1-\delta)^k} \int_0^{\infty} \left(1-\delta e^{-x^{\beta}}\right)^{k-1} \left(\beta\delta e^{-x^{\beta}} x^{\beta-1}\right) dx,$$

$$= \frac{k}{1-(1-\delta)^k} \left[\frac{\left(1-\delta e^{-x^{\beta}}\right)^k}{k}\right]_0^{\infty},$$

$$= \frac{k}{1-(1-\delta)^k} \left(\frac{\left(1-(1-\delta)^k\right)^k}{k}\right)$$

$= 1.$

This shows that $f(x,\alpha,\beta,k)$ is the p.d.f. of random variable X of continuous type. This proves Theorem 4.

REMARK (4.1): Weibull distribution is the limiting case of Theorem 4.To see this, we let $\delta \to 0$ in Theorem 4 and note that

$$\lim_{\delta\to 0} \frac{k\delta}{1-(1-\delta)^k} = 1 .$$

so that

$$f(x,\beta) = \beta e^{-x^{\beta}} x^{\beta-1}$$

$$= e^{-x^{\beta}} \frac{d}{dx}\left(x^{\beta}\right), \qquad x>0,$$

$$= 0, \qquad x \le 0,$$

is the p.d.f.of random variable X of continuous type.

Replacing x by $\frac{x-\nu}{\alpha}$, it follows that

$$f(x,\alpha,\beta,\nu) = \beta e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}} \frac{d}{dx}\left(\frac{x-\nu}{\alpha}\right)^{\beta}, \qquad \frac{x-\nu}{\alpha} > 0$$

$$= 0, \qquad \frac{x-\nu}{\alpha} \le 0.$$

Equivalently

$$f(x,\alpha,\beta,\nu) = \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x-\nu}{\alpha}\right)^{\beta}}, \qquad x>\nu,$$

$$= 0, \qquad x \le \nu,$$

is a p.d.f. of X, which is clearly density of Weibull distribution.

VII. CONCLUSIONS

In this paper the authors conclude the following:

- ✓ Weibull distribution is a special case of k-Generalized Exponential Distribution.
- ✓ Also if δ tends to 1, then our newly introduced 3 parameter extension of Generalized Exponential Distribution (GED) reduced to classical Exponential Distribution.
- ✓ Further if δ tends to 0, then our lastly proved more general result leads to Weibull distribution.
- ✓ The moment generating functions (m.g.f's) obtained in this paper generalize the classical moment generating functions (m.g.f's) of the given distributions.

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