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Weak Form Of Slight Continuity

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***Abstract:** In this paper we apply the notion of b*-open sets to study a class of functions called slightly b*-continuous functions. Also the concept of slightly b*-closed graphs and b*-compactness is investigated. Some characterizations and properties of these concepts are presented.*

I. INTRODUCTION

The notion of continuity has shown its significance not only in topology but also in other branches of mathematics. So different types of weak and strong continuities have been introduced over years. Levine [9] began this study by introducing the concept of semi-open sets. Many mathematicians have worked in this direction. Later on Jain [7] made his contribution by giving the notion of slightly continuous functions. Many weak forms of slightly continuous functions have been discussed. For instance slightly semi-continuous [15] slightly β- continuous [13], slightly γ-continuous [6], slightly ω- continuous [14] and slightly b-continuous functions [4] have been explored. In this paper we investigate some weak forms of open sets namely b*open sets. By means of these sets we investigate some concepts of classes of functions, namely slightly b*-continuous functions, slightly b*-closed graphs and b*-compactness.

II. NOTATION, DEFINITIONS AND PRELIMINARIES

Throughout the paper spaces mean topological spaces and f: (X, τ) → (Y, σ) or (f: X → Y) mean a function of a space (X, τ) into a space (Y, σ). Let X be a topological space and B ⊂ X. cl (B) and int (B) are used to denote the closure of B and interior of B respectively. B is called clopen set of X if it is both open and closed in X.

DEFINITION 2.1: A subset B of X is called δ*-open [16] if for each x ∈ B there exists an open and closed set V of X such that x∈V⊂B.

DEFINITION 2.2: A subset B of X is called α-open [12] if B⊂int(cl(intB))).

DEFINITION 2.3: A subset A of a topological space (X, τ) is called a b∗ closed set[10] if int(cl(A)) ⊆ U , whenever A ⊆ U and U is b-open.

DEFINITION 2.4: A function f is said to be slightly continuous [7] if for every clopen subset Uof Y, the set f^{-1} (U) is open in X.

DEFINITION 2.5: f is slightly b*-continuous at a point x ∈ X if for each clopen subset U of Y containing f(x), there exists b*-open subset V in X containing x such that f (V) ⊂U. f is said to be slightly b*-continuous if it is slightly b*-continuous at each point of X.

DEFINITION 2.6: function f: X → Y is called

(i) b*-irresolute if for every b* -open subset U of Y, f^{-1}(U) is b* -open in X,

(ii) b*-open if for every b* -open subset V of X, f(V) is b* -open in Y,

(iii) Weakly b* -continuous if for each point x∈X and each open subset U of Y containing f(x), there exists b*-open subset V in X containing x such that f(V) ⊂ cl(U),

(iv) Contra b* -continuous if f^{-1} (H) is b* -open for each closed set H of Y.

DEFINITION2.7: A space X is said to be

(1) Ultra Hausdorff [17] if every two distinct points of X can be separated by disjoint clopen sets,

(2) Extremely disconnected [3] if the closure of every open set of X is open in X,

(3) Locally indiscrete [11] if every open set of X is closed in X,

(4) 0-dimensional [18] if its topology has a base consisting of clopen sets,

DEFINITION 2,8: A space X is said to be

(1b*-T_0 (resp. b*-T_1) if for each x, y ϵ X such that x≠ y,there exists b*-open set containing x but not y, or (resp. and) b*-open set containing y but not x;

(2) b*-T_2 if for each x, y ϵ X such that x ≠ y. there exist disjoint b*–open sets V and U such that x ϵV and y ϵU;

(3) b*-regular if for each closed set H of X and each point x ∉ H, there exist disjoint b*-open sets U and V such that H⊂U and xϵV;

(4) b*-normal if for every pair of disjoint closed sets H_1 and H_2 of X there exist disjoint b*-open sets U and V such that H_1⊂U and H_2⊂V;

(5) b*-compact if every b* -open cover of X has a finite subcover;

(6) cl-compact if every cl-open cover of X has a finite subcover;

(7) b*-connected A space X is said to be b*-connected between subsets A and B provided there is no b*-clopen set H for which A⊂ H and H∩ B = ϕ.

DEFINITION 2.9: A subset B of X is said to be b* *-closed if for each x ϵX−B there exists a b*-clopen set V containing x such that V∩B= ϕ. .

DEFINITION 2.10: A function f: X →Y is said to be set b*- connected if whenever X is b*-connected between A and B, then f(X) is connected between f (A) and f (B) with respect to the relative topology on f (X).

DEFINITION 2.11: For a function f: X →Y, the graph G(f)= {(x, f(x))|x ϵX} is called slightly b*-closed graph if for each (x, y) ϵ ((X×Y) –G(f)), there exist V ϵ b*(X, x) and a clopen set U containing y such that (V× *U*) ∩ G (f) = ϕ.

III. SOME CHARACTERIZATIONS

THEOREM 3.1 The following properties hold for a function f: X → Y:

(1)f is slightly b* -continuous;

(2)For every clopen set U⊂Y, f^{-1} (U) is b*-open;

(3)For every clopen set U ⊂ Y, f^{-1} (U) is b* -closed;

(4)For every clopen set U⊂ Y, f^{-1} (U) is b* -clopen;

(5) For every δ*-open set U⊂ Y, f^{-1} (U) is b* -open;

(6)For every δ* -closed set U ⊂ Y, f^{-1} (U) is b* -closed.

PROOF. (1) ⟹ (2): Let U be a clopen subset of Y and let x ϵ f^{-1} (U). Since f is slightly b* -continuous, by (1) there exists b* -open set Vx in X containing x such that f (Vx) ⊂ U; hence Vx ⊂f^{-1} (U). We obtain that f^{-1} (U) =∪ [{Vx |x ϵ f^{-1} (U)].Thus f^{-1} (U) is b* –open.

(2) ⟹ (3): Let U be a clopen subset of Y. Then Y \U is clopen. By (2) f^{-1} ((Y \ U) = X \ f^{-1} (U) is b* -open. Thus f^{-1} (U) is b* -closed.

(3) ⟹ (4): It can be proved easily.

(4) ⟹ (5): Let Ube δ*-open open set in Y and let x ϵ f^{-1} (U). Then f(x) ϵ U. Since U is δ*-open there exists a F ϵ CO(Y) such that f(x) ϵ F ⊂ U. This implies that x ϵ f^{-1} (F) ⊂ f^{-1} (U). By (4) f^{-1} (F) is b* -clopen. Hence f^{-1} (U) is b*-neighbourhood of each of its points. Consequently, f^{-1} (U) ϵ b*O(X).

(5) ⟹ (6): It is clear from the fact that the complement of δ*-closed is δ*-open.

(5) ⟹ (1): It is clear from the fact that every clopen set is δ*-open.

COROLLARY3.2 Let (X, τ) and (Y, σ) be topological spaces. The following statements are equivalent for a function f: X →Y

(1) f is slightly b* -continuous;

(2)for every x ϵ X and each clopen set U⊂ Y containing f(x), there exists V ϵ b*O(X, x) such that f(V) ⊂ U .

IV. SEPARATION AXIOMS AND GRAPH FUNCTION

THEOREM 4.1. Let Y be a 0-dimensional space and f: X → Y be a slightly b*-continuous injection. Then the following properties hold:

If Y is T_1 (resp. T_2), then X is b*- T_1 (resp.b* -T_2).

If f is either open or closed, then X is b* -regular.

If f is closed and Y is normal, then X is b* -normal.

PROOF

(1) We prove only the second statement, the proof for the first being similar. Let Y be T_2. Since f is injective, for any pair of distinct points x_1, x_2 ϵ X, f(x_1) ≠ f(x_2). Since Y is T_2, there exist open sets U_1, U_2 in Y such that f(x_1) ϵ U_1, f(x_2) ϵ U_2 and U_1 ∩U_2 = ϕ. Since Y is a 0-dimensional space, there exist V_1, V_2 ϵ CO(Y) such that f(x_1) ϵ V_1 ⊂ U_1 and f(x_2) ϵ V_2 ⊂ U_2. Consequently x_1 ϵ f^{-1}(V_1) ⊂ f^{-1} (U_1), x_2 ϵ f^{-1}(V_2) ⊂ f^{-1}(U_2), and f^{-1}(V_1) ∩ f^{-1}(V_2) = ϕ. Since f is slightly b*-continuous, f^{-1}(V_1) and f^{-1}(V_2) are b*-open sets.It follows that X is b*–T_2.

(2) First suppose f is open. Let x ϵ X and V be an open set containing x. Then, f(x) ϵ f (V) which is open in Y because of the openness of f. Moreover, 0-dimensionality of Y gives the existence of a UϵCO(Y) such that f(x) ϵU⊂f (V). We obtain x ϵ f^{1}(V) ⊂U,for f is injective. Since f is slightly b*-continuous, so f^{-1}(V) is an b*-clopen set in X by 3.1 and hence x ϵ f^{1} (V) = Cl_{b*} (f^{1}(V)) ⊂ U. This implies that X is b*-regular. Now suppose f is closed. Let x ϵ X and H be a closed set of X such that x ∉ H. Then, f(x) ∉ f (H) and f(x) ϵY - f (H) which is an open set in Y since f is closed. But Y is 0-dimensional and there exists a clopen set U in Y such that f(x) ϵU⊂Y –f (H). Since f is slightly b*-continuous, we have x ϵ f^{-1} (U) ϵ b*CO(X) and H ⊂ X − f^{-1}(U) ϵ b*CO(X). Therefore X is b*-regular.

(3) Let H_1 and H_2 be any two closed sets in X such that H_1 ∩ H_2 = ϕ. Since f is closed and injective, we have f (H_1) and f (H_2) are two closed sets in Y with f (H_1) ∩ f (H_2) = ϕ. By normality of Y, there exist two open sets U and V in Y such that f (H_1) ⊂U, f(H_2)⊂V and U∩V = ϕ. Let y ϵ f (H_1), then yϵU. Since Y is 0-dimensional and U is open in Y, there exists a clopen set Uy such that y ϵ U y ⊂U. Then f(H_1) ⊂∪ {Uy|Uy ϵ CO(Y), y ϵf(H_1)} ⊂ U, and thus H_1 ⊂ ∪ [{f^{-1}(Uy)|Uy ϵ CO(Y), y ϵ f(H_1)}⊂ f^{-1} (U). Since f is slightly b*-continuous, f^{-1}(Uy) is b*-open for each Uy ϵ CO(Y) so that K = ∪ {f^{-1}(Uy) |yϵ f (H_1)} is b*-open in X and H_1 ⊂ K ⊂ f^{-1}(U). Similarly, there exists an b*-open set F in X such that H_2 ⊂ F

$\subset f^{-1}$ (V) and K∩F⊂f^{-1}(U∩V) =ϕ. This shows that X is b*-normal.

REMARK:4.2 $T_1 \Longrightarrow$ b*- T_1.

But this implication is not reversible as is clear from following example:

Let X=$\{a, b, c\}$, with $\tau = \{X, \emptyset, \{a\}\{b, c\}\}$.

Then (X, τ) is b*- T_1 but not T_1.

THEOREM 4.3. Let f: X → Y be a function and G (f) the graph function of f, defined by G (f) (x) = (x, f(x)) for every x ϵ X. Then G (f) is slightly b*-continuous if and only if f is slightly b*-continuous.

PROOF. Let U ϵ CO(Y), then X × U is clopen in X × Y. Since G (f) is slightly b*-continuous, then f^{-1}(U) = G (f) $^{-1}$(X × U) ϵ b*O(X). Thus, f is slightly b*-continuous. Conversely, let x ϵ X and H be a clopen subset of X ×Y containing G (f) (x). Then H ∩ ({x} ×Y) is clopen in {x} ×Y containing G(f)(x). Also {x} × Y is homeomorphic to Y. Hence {y ϵ Y |(x, y) ϵ H} is a clopen subset of Y. Since f is slightly b*–continuous, ∪ {f^{-1}(y)|(x, y) ϵ H} is a b* -open set of X. Further x ϵ∪ {f^{-1}(y)|(x, y) ϵ H} ⊂ G (f) $^{-1}$(H). Hence G (f)$^{-1}$(H) is b*-open. It follows that G (f) is slightly b*-continuous.

THEOREM 4.4. Let f : X → Y be slightly b*-continuous and Y be ultra Hausdorff and the product of two b*-open sets is b* -open, then (1) The graph G(f) of f is b* *-closed in the product space X × Y . (2) The set {(x_1, x_2)|f(x_1) = f(x_2)} b* *-closed in the product space X × X.

PROOF. (1) Let (x, y) ϵ (X×Y) −G (f). We have f(x) ≠ y. Since Y is ultra Hausdorff, there exist clopen sets W and V such that y ϵ W and f(x) ϵ V and W ∩ V = ϕ. Since f is slightly b* -continuous, there exists b*-clopen set U containing x such that f (U) ⊂ V. Therefore, we obtain W ∩ f (U) = ϕ and hence (U ×W) ∩ G (f) = ϕ and U ×W is an b*-clopen set of X ×Y. It follows that G (f) is b* *-closed in X × Y.

(2) Set B= {(x_1, x_2) |f(x_1) = f(x_2)}. Let (x_1, x_2) ∉ B then f(x_1) ≠ f(x_2). Since Y is ultra Hausdorff, there exist U_1, U_2 ϵ CO(Y) containing f(x_1), f(x_2) respectively, such that U_1 ∩U_2 = ϕ. Since f is slightly b*-continuous, there exist b*-clopen sets V_1,V_2 of X such that x_1 ϵ V_1, f(V_1) ⊂ U_1 and x_2 ϵ V_2, f(V_2) ⊂U_2; hence f(V_1) ∩ f(V_2) = ϕ. Thus (x_1, x_2) ϵV_1 × V_2 and (V_1 × V_2) ∩ B = ϕ. Moreover V_1 × V_2 is b*-clopen in X × X containing (x_1, x_2).It follows that B is b* *-closed in the product space X × X.

V. b*-CONNECTEDNESS

THEOREM5.1. If f: X → Y is slightly b* continuous surjection and X is b*-connected, then Y is connected.

PROOF. Assume that Y is a disconnected space. Then there exist nonempty disjoint open sets V and U such that Y = V ∪ U. Therefore, V and U are clopen sets in Y. Since f is slightly b*-continuous, f^{-1}(V) and f^{-1} (U) are b*-clopen in X. Moreover, f^{-1}(V) and f^{-1}(U) are disjoint and X = f^{-1}(V)∪ f^{-1}(U). Since f is surjective, f^{-1}(V) and f^{-1}(U) are nonempty. Therefore, X is not b*-connected. Which is a contradiction and so Y is connected.

THEOREM5.2. A function f: X →Y is set b*-connected if and only if f^{-1}(H) is b*-clopen for every clopen subset H of f(X) (with respect to the relative topology on f(X)).

PROOF. Assume that H is a clopen subset of f(X) with respect to the relative topology on f(X). Suppose that f^{-1}(H) is not b*-closed in X. Then there exists x ϵ X - f^{-1}(H) such that for every b*-open set V with x ϵ V we have V∩ f^{-1}(H) ≠ ϕ. We claim that the space X is set b*-connected between x and f^{-1}(H). Suppose there exists an b*-clopen set B such that f^{-1}(H) ⊂ B and x ∉ B. Then x ϵ X − B⊂ X – f^{-1} (H) and clearly X −B is an b*–open set containing x and disjoint from f^{-1}(H), this contradiction implies that X is set b* -connected between x and f^{-1}(H).Since f is set b*-connected, f(X) is connected between f(x) and f(f $^{-1}$(H)). But f (f^{-1}(H)) ⊂ H which is clopen in f(X) and f(x) ∉ H, which is a contradiction. Therefore f^{-1}(H) is b*-closed in X and similarly it can be shown that f^{-1}(H) is b*-open.

Next assume that X is b*-connected between A and B and also f(X)is not connected between f(A) and f(B) (in the relative topology on f(X)).Thus there is a set H ⊂ f(X) that is clopen in the relative topology on f(X)such that f(A) ⊂ H and H ∩ f(B) = ϕ . Then A ⊂ f^{-1}(H), B ∩ f^{-1}(H) = $\phi$and f^{-1}(H) is b*-clopen, which implies that X is not b*-connected between A and B. It follows that f is set b*-connected.

COROLLARY5.3. Every slightly b*-continuous surjection is set b*-connected.

THEOREM 5.4. Every set b*-connected function is slightly b*–continuous

PROOF. Assume f: X → Y is set b*-connected. Let H be a clopen subset of Y. Then H ∩ f(X) is clopen in the relative topology on f(X). Since f is set b*′ -connected, by Theorem 4.4, f (H) $^{-1}$ = f^{-1}(H∩ f(X)) is b*-clopen in X.

COROLLARY 5.5. A surjective function is slightly b*-continuous if and only if it is set b*–connected.

VI. SLIGHTLY b*-CLOSED GRAPHS

LEMMA 6 .1 A function f: X⟶ Y has slightly b*-closed graph if and only if for each x ϵ X and y ϵ Y such that f(x) ≠ y, there exists V∈ b*(X, x) and a clopen set U containing y such that f (V) ∩U =∅.

PROOF . It can be obtained immediately from Definition.

THEOREM 6.2 Let f: X⟶ Y be slightly b*-continuous and Y be ultra Hausdorff, then G (f) is slightly b* -closed.

PROOF. Assume that (x, y) is any point of (X×Y) – G (f). Then y ≠ f(x). Since Y is ultra Hausdroff so, there exist clopen sets K_1 and K_2 in Y such that y ϵ K_1, f (x) ϵ K_2 and K_1∩K_2=∅. Since f is slightly b*–continuous there exists U∈b * o(X) such that f(U)⊂cl(K_2) ⇒ $f(U)$∩K_1=∅.It follows from lemma 5.1 G(f) is slightly b*-closed.

THEOREM6.3 let f: X→Y have a slightly b*–closed graph G (f) If f is injective, then X is b*-T_1

PROOF: Let x and y be any two distinct points of X. Then we have (x, f(y)) ϵ ((X×Y) –G (f)). Since G (f) is slightly b*–closed graph there exists V∈b* (X, x) and a clopen set U containing f(y) such that f (V) ∩U= ∅.Therefore V∩ f^{-1}(U) = ∅ and hence y ∉ V. It follows that X is b*-T_1.

THEOREM: 6.4 Let f: X→Y have a slightly b*–closed graph G (f). If f is surjective, b*-open, then Y is b*-T_2.

PROOF: Let y_1 and y_2 be any distict points of Y. Since f is surjection, f(x)=y_1 for some x ∈ X and (x,y) ϵ ((X×Y) – G(f)).Since f has the slightly b*-closed graph, therefore there exists V ∈b* (X, x) and a clopen set U containing y_2 such that ((V×U) ∩ G(f)) = ∅. Hence f(V) ∩U = ∅. Since f is b*-open, f (V) is b*-open such that f(x) =y_1 ϵf (V). Therefore Y is b*-T_2.

VII. COMPACTNESS

THEOREM7.1. Let f: X→Y be slightly b*-continuous and H be b*-compact relative to X, then f (H) is cl-compact relative to Y.

PROOF: Let {G_α|αϵΔ} be any cover of f (H) by clopen sets of Y. By Theorem 3.1 since f is slightly b*-continuous, {f^{-1}(G_α) |α ϵΔ} is a cover of H by b*-open sets of X. Since H is b*-compact relative to X, there exists a finite subset Δ_0 of Δ such that H⊂ ∪ {f^{-1}(G_α) | αϵ Δ_0}. Therefore f (H) ⊂ ∪ {G_α | α ϵ Δ_0}.This shows that f(H) is cl-compact relative to Y.

COROLLARY 7.2. Let f: X→Y be slightly b*-continuous surjection and H be b*-compact relative to X, then Y is cl-compact relative to Y.

THEOREM7.3 Let f: X→ Y have a slightly b*–closed graph, Then f (H) is δ*-closed in Y for each subset H which is b*-compact relative to X.

PROOF: Suppose that y∉ $f(H)$.Then (x,y) ϵ ((X×Y) –G(f)) for each x ϵH.Since G(f) is slightly b*-closed graph , there exists V∈b* (X, x) and a clopen set U_x of Y such that f(V_x) ∩U_x= ∅. The family {V_x | x ϵ } is a cover of H by b*-open sets of X.Since H is b* -compact relative to X, there exists a finite subset H_0 of H such that such that H ⊂ { V_x | x ϵ H_0 }. Set U=∩ {U_x | x ϵ H_0} then U is clopen set in Y containing y. Therefore we have f(H) ∩U ⊂ ∪{f(V_x) | x ϵ H_0}∩ U⊂∪{f(V_x) ∩ U | x ϵ H_0}=∅ . It follows that y ∉ δ*-cl(f(H)). It follows that f(H) is δ*-closed in Y.

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