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The Action Of Dicyclic Group Q_{12} Of Order 24 In Pitch Analysis

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Abstract: ***We present the symmetric properties of the non-abelian dicyclic group Q_{12} using group action and topology. The resulting symmetric group is generated by the symmetric operations transposition (t) and inversion (f) of a regular 12-gon. Each of the operations generated 12 symmetry elements which are classified as two distinguished copies of Dicyclic group of order 24. As the sequence of pitches which form musical melody can be transposed or inverted, we therefore modeled the musical properties in terms of an action of the dicyclic group of order 24. We illustrate both geometrically and algebraically how these properties are used to analyze works of music in diverse area of algebra.***

Keywords: Symmetry group, Orbit, Stabilizer, Conjugacy, Centralizer, Music.

I. INTRODUCTION

If knowledge of group structures can influence how we see crystals, perhaps it can influence how we hear music as well [1]. Our concerned in this work is how we can use the group structure of Dicyclic group Q_{12} of order 24 and its Centralizer to interpret music as used by Babbit [2] in musical actions of Dihedral groups. The dicyclic group of order 24 is a group of symmetries of regular 12-gon generated by two symmetric elements namely; rotation r, and flipping f, subject to the following relations;

$r^{12} = 1 \qquad s^2 = 1 \qquad rsr = s$

The group operation is function composition, satisfying all properties of group theory except that $\underline{Q_{12}}$ is not abelian.

Of the 45 non-abelian groups, the dihedrals (D_n) and the dicyclic (Q_{2n}), of order $2n$ and $4n$ respectively, formed the simplest sequences. The generated sequence consists of transposition (rotation) and inversion (flipping) which we consider as the compositional techniques for the musical action of dicyclic group of order 24. As the sequence of pitches which form a musical melody can be transposed or inverted, music theorists have modeled musical transposition and inversion in terms of an action of the dicyclic group of order 24 [3]. We illustrate both geometrically and algebraically how these properties are used to analyze works of music in diverse areas of algebra, also described by Samaila [4] as dual with the group action of Q_{12} which come to the attention of music theorist in the past two decades. That is, the P (parallel operation), L (leading tone exchange operation) and R (relative operation) operations of the 19th-century music theorist Hugo Riemann.

Bach often used diatonic transposition and inversion, which can be viewed as mod 7 transpositions and inversions after identifying the diatonic scale with Z_7. However, many contemporary composers intensively use mod 12 transpositions and inversions [5, 6]. The Pitch classes are translated to the group of integers modulo 12 (Z_{12}) [7], where some music theorist found it easy to described the pitch classes. These permit us to easily use algebra for modeling musical events. A transposition moves a sequence of pitches up or down when singers decide to sing a song in a higher register. In this process, the singers achieved it by transposing the melody. An inversion, on the other hand reflects a melody about a fixed axis, just as the face of a clock can be reflected about the 0-6 axis or 3-9 axis. Often, musical inversion turns upward melodic motions into downward melodic motions. One can hear both transpositions and inversions in many fugues, such as Bernstein's "Cool" fugue from West Side Story or in Bach's Art of Fugue [1]. We will mathematically see that these musical transpositions and inversions are the symmetries of the regular 12-gon.

II. THE DICYCLIC GROUP Q_{12} OF ORDER 24

Algebraically, the *dicyclic group* of order *24* is the group of symmetries of regular 12-gon generated by two symmetry operations t (through an angle of 30^o each) and f (along the lines of symmetries) subject to the relations $t^{12} = 1$, $f^2 = 1$, $tft = f$ and $ftf = t^{-1}$. The operation t (transposition) generated *12* symmetry elements through an angles multiple of 30^o each [8]. Denoting the set by H, we have

$H = \{t, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9, t^{10}, t^{11}, t^{12} = 1\}$

Where t is taken to be the first anti-clockwise transposition of the regular 12-gon, i.e. $t = (1\ 12\ 11\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2)$. The set H of all transpositions of Q_{12} formed a group called Cyclic group with generator t, i.e. $H = \langle t \rangle$. The order of H is *12* while the order of each element $t^n \epsilon H$ is n if n is the least positive integer for which n = 0 mod12 and m if m is the least positive integer for which $n+m$=0 mod12, since $t^{12}=1=t^{0mod12}$, where 1 is the identity transformation, taking the regular 12-gon back to its original position.

The set of all inversions of Q_{12} are expressed as products of f and t given by $S = \{f, tf, t^2f, t^3f, t^4f, t^5f, t^6f, t^7f, t^8f, t^9f, t^{10}f, t^{11}f\}$, where f is the reflection about the vertical axis of the regular 12-gon, i.e. $6\leftrightarrow12$ given by $f = (1\ 5)(2\ 4)(6\ 12)(7\ 11)(8\ 10)$ with the property that $t^nf \cdot t^mf = t^{(n-m)mod12}$, and each element is its own inverse. Unfortunately for the set S, it does not form a group. Though each element is its own inverse, but $\forall\ \rho\in S$, $\rho\cdot\rho^{-1}=1\notin S$. Also, all other group axioms are not satisfied since S is not closed. With these two sets, we have $Q_{12}=H+S$ where $H\cap S=\Phi$, the empty set. Thus, $Q_{12} = \langle t, f \mid t^{12}=1=f^2\rangle$.

Obviously, H is a normal subgroup of Q_{12} since $[Q_{12}:H]=2$ by Lagrange's theorem [9]. The subgroup $H_1=\{1, t^6\}$ is also a normal subgroup of Q_{12} since $\rho\mu\rho^{-1}\in H_1\ \forall\ \rho\in Q_{12}$ and $\forall\ \mu\in H_1$.

A. GROUP ACTIONS, ORBITS AND STABILIZERS IN Q_{12}

Recall that a left action of a group H on a set S associates to each $t\in H$ and $f\in S$ an element $t\cdot f$ of S in such a way that $t\cdot(h\cdot f) = (th)\cdot f$ and $1\cdot f = f$ for all $t,h\in H$ and $f\in S$, where 1 denotes the identity element of H. Hence, the left action of the group H on the set S described above associate to each $t\in H$ and $f\in S$ an element $t\cdot f$ of S since each $t\cdot f\in S\ \forall\ t\in H$ and all $f\in S$. That is, each element of the group H described the left action of H on the set S, where $H\cdot S=S$.

Given a left action of a group H on the set S, the orbit of an element f of S is the subset $\{t\cdot f : t \in H\}$ of S. This shows that the Orbit of each element $f\in S$ is the whole set S, of order 12.

Furthermore, the stabilizer of $f\in S$ is the subgroup $\{t\in H : t\cdot f = f\}$ of H. We observed that the Stabilizer of each element $f\in S$ is the trivial subgroup $H_2=\{1\}$ of H, where 1 is the identity element of H.

LEMMA 2.1.1: Let G be a finite group which acts on a set X on the left. Then the orbit of an element x of a set X contains [G:H] elements, where [G:H] is the index of the stabilizer H of x in G [10].

PROOF: There is a well-defined function $\Phi:G/H\rightarrow X$ defined on the set G/H of left cosets of H in G which sends gH to $g\cdot x$ for all $g\in G$. Moreover, this function is injective, and its image is the orbit of x. Hence, the result follows.

REMARK 2.1.2: The orbit of each element $f\in S$ described above is found to be the set S itself of order *12*, and the set of stabilizer H_2 of each element $f\in S$ is a singleton. Now, since order of H is 12, we have by Lagrange's theorem $[H:H_2]=|H|/|H_2|=12$ as required.

B. THE CONJUGACY CLASS AND CENTRALIZER OF Q_{12}

Two elements g and h of the group Q_{12} are said to be conjugate if $g=jhj^{-1}$ for some $j\in Q_{12}$. We observed that in the subgroup H of Q_{12}, the relation of Conjugacy is reflexive, i.e. $tht^{-1}=h$ for all $h\in H$ and all $t\in H$. We also observed the following Conjugacy classes in Q_{12}:

(1) We have $ft^nf^{-1}=t^{12-n}$ for all positive integers n, where $1 \le n \le 12$.

(2) For all integers $1 \le n \le 12$ and $0 \le m \le 12$, we have $t^n(t^mf)t^{-n} = t^{(m+2n)mod12}f$. (Not that $(t^n)^{-1} = t^{-n}$).

The relation for which $n = 6$ (and $n = 12$) shows that $t^6(t^mf)t^{-6} = t^mf$ for all m, showing the reflexivity of Q_{12}. Again, $tht^{-1} = g$ implies $ygy^{-1} = h$ for some $y\in Q_{12}$, that is, the relation of Conjugacy is symmetric. And finally, we observed that if $tht^{-1} = g$ and $ygy^{-1} = j$, then $xhx^{-1} = j$, for some $t,y,x\in Q_{12}$ i.e. the relation of Conjugacy is transitive, and hence, an *equivalence relation*.

It is also found that the Conjugacy $t^n(t^mf)t^{-n} = t^{(m+2n)mod12}f$ is partitioned in to two classes for all m. One class is for $1\le n\le6$ and the other is for $7\le n\le12$ and there exist a one-to-one onto mapping between the two classes where $t^1(t^mf)t^{-1} = t^7(t^mf)t^{-7}$, $t^2(t^mf)t^{-2} = t^8(t^mf)t^{-8}$, $t^3(t^mf)t^{-3} = t^9(t^mf)t^{-9}$ and so on, and $t^6(t^mf)t^{-6} = t^{12}(t^mf)t^{-12}$.

Furthermore, we recall that the centralizer of the subgroup H in a group Q_{12} is the set of elements of Q_{12} which commute with all elements of H, namely $C(H)_{Q12} = \{g \in Q_{12} \mid gt = tg$ for all $t \in H\}$. Hence, the centralizer of the subgroup H of the group Q_{12} is the subgroup H itself. Again the center of the group Q_{12} is the subgroup of Q_{12} defined by $Z(Q_{12}) = \{g\in Q_{12} : gh = hg\ \forall\ h\in Q_{12}\}$. Thus, $Z(Q_{12}) = \{1, t^6\}$, where 1 is the identity element of Q_{12}.

III. THE GROUP OF INTEGERS MODULO 12

The set Z_m of integers modulo m is a group with respect to addition while $(z_m,\cdot)$ is not. Our aim here is to describe the nature of $(Z_{12},-)$ instead of $(Z_{12},+)$ using the table below.

$(Z_{12},-) = \{1,2,3,4,5,6,7,8,9,10,11,0\} = (Z_{12},+)$

-	1	2	3	4	5	6	7	8	9	10	11	0
1	0	11	10	9	8	7	6	5	4	3	2	1
2	1	0	11	10	9	8	7	6	5	4	3	2
3	2	1	0	11	10	9	8	7	6	5	4	3
4	3	2	1	0	11	10	9	8	7	6	5	4
5	4	3	2	1	0	11	10	9	8	7	6	5
6	5	4	3	2	1	0	11	10	9	8	7	6
7	6	5	4	3	2	1	0	11	10	9	8	7
8	7	6	5	4	3	2	1	0	11	10	9	8
9	8	7	6	5	4	3	2	1	0	11	10	9

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10	9	8	7	6	5	4	3	2	1	0	11	10
11	10	9	8	7	6	5	4	3	2	1	0	11
0	11	10	9	8	7	6	5	4	3	2	1	0

Table 1: Table of (Z_{12},-)

Obviously, (Z_{12},-) is closed since $x - y \in Z_{12}$ for all $x,y \in Z_{12}$. But (Z_{12},-) is not associative. To see this, $(2 – 3) – 4 = 11 – 4 = 7$ and $2 – (3 – 4) = 2 – 11 = 3$, i.e. $(2 – 3) – 4 \neq 2 – (3 – 4)$. Again, the identity element 0 does not satisfy $x – 0 = 0 – x = x$ for all $x \in Z_{12}$. Hence, (Z_{12},-) is not a group.

IV. THE PITCH CLASSES

Any two pitches that differ by a whole number of octaves i.e., an interval between two notes consisting of eight notes inclusive or seven steps on the diatonic scale, sounds alike as noticed by ancient Greeks [1]. Thus we identify any two such pitches, and speak of pitch classes arising from this equivalence relation. Like most modern music theorists, we used well proportion tuning in order to divide the octave into twelve pitch classes giving below.

A	AG BI	B	C	CG DI	D	DG EI	E	F	FG GI	G	GG AI	A

Table 2: The 12 pitch classes

The smallest interval of the diatonic scale, half of a whole tone or interval between two consecutive pitch classes is called a semitone, where the symbol G means to move up a semitone while the symbol I means to move down a semitone [11].

The instance of a pitch having two letter names, i.e. having same pitch in tempered scale, describes musical notes that appear differently in a score but have the same pitch in a tempered scale e.g. on the piano. This is called enharmonic equivalence. In other scales or on other instruments, enharmonic notes may actually have different pitches.

Music theorists have found it useful to translate pitch classes to integers modulo 12, where 0 is taking to be C as seen in Figure 1 below. Addition mod 12 is read in the clockwise sense of the clock, for example, 2+5=7 mod12, 9+11=8 mod12 and so on. While subtraction mod 12 is read in the anti-clockwise sense of the clock as 1-6=7 mod12, 4-11=5 mod12, 9-3=6 mod12 and so on which can as well checked in table 1 above.

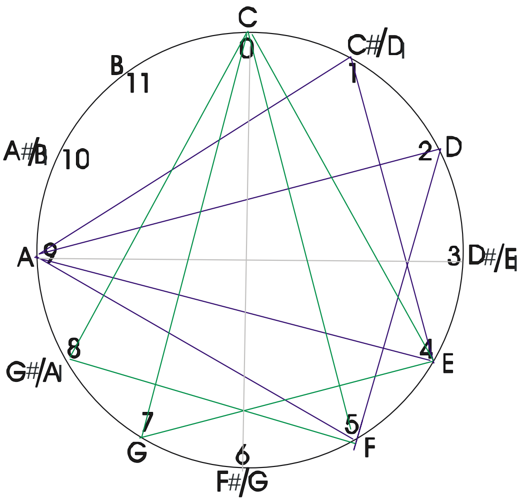

Figure 1: The musical clock

The musical interval from one pitch class to another can also be determine, for example, the interval from C# to A is eight semitones. The description of pitch classes in terms of Z_{12} is what enables us to use algebra for modeling musical events [7].

V. TRANSPOSITION AND INVERSION

The popular musical form used by composers in which a theme is first stated, then repeated and varied with accompanying contrapuntal lines called *fugue*, is a musical composition popular in the 18th century associated with J. S. Bach. Throughout the ages, the composers have drawn on the musical tools of transposition and inversion. Such a composition contains a principal melody known as the subject; as the fugue progresses, the subject typically will recur in transposed and inverted forms [1].

We shall now mathematically define a transposition by an integer n mod12 as a function t^n mapping Z_{12} onto Z_{12} and inversion $t^n f$ about n as a function $t^n f$ mapping Z_{12} onto Z_{12}, for n=1, 2, …, 12 defined by

$t^n: Z_{12} \rightarrow Z_{12}$ such that $t^n(x)=x+n \bmod 12$ and

$t^n f: Z_{12} \rightarrow Z_{12}$ such that $t^n f(x)=-x+n \bmod 12$.

It can be seen clearly that these transpositions and inversions are well defined in the musical clock giving in figure 1 above. Each transposition t^n is taking through an angle of $30^o n$ consecutively with n in the clockwise sense, and $t^n f$ corresponds to the reflection of the clock along the 0-6 axis (i.e. vertical flipping) at the point where n=12. The fact that $t^n \cdot t^m = t^{(n+m)mod12}$ and $t^n \cdot t^m f = t^{(n+m)mod12} f$ shows that the two elements t^n and $t^n f$ generate the Dicyclic group of order 24 as discussed in section 2.0. The group is also called t/f – group.

The following pattern of compositions are also observed in the two sets H and S of transpositions and inversions respectively that formed the group of symmetries of regular 12-gon, Q_{12}.

$t^n f \cdot t^m = t^{(n-m)mod12} f$

$t^n f \cdot t^m f = t^{(n-m)mod12}$

Using the values of t and f given in section 2.0 above, all these relations can be easily verified by product of permutations which is function composition.

A. MAJOR AND MINOR TRIADS

The combination of musical chord consisting of three notes, especially, a chord made up of a tonic, a third, and a fifth used to play at the same time in any popular music. The integers modulo *12* are used to define major and minor triads [12], which are considered as the objects upon which the dicyclic group of order *24* act. A triad consists of three simultaneously played notes as mentioned above. A major triad consists of a root note, a second note which is 4 semitones above the root, and a third note 7 semitones above the root. An example can be seen in A-major triad which consists of {9, 1, 4} = {A, C#, E} and is represented as a chord polygon in Figure 1 above [7]. Also, any major triad is a subset of the pitch-class space Z_{12}, and since transpositions and inversions act on Z_{12}, it also acts on any major triad. Also, the action of $t^n f$ when $n = 9$ on A-major triad can be deduced from Figure 1, where the resulting triad is a minor triad.

A minor triad usually denoted by lower case letters also consists of a root note, a second note is 8 semitones above the root, and a third note 5 semitones above the root. For example, the *d*- minor triad consists of {9, 5, 2} = {A, F, D} and its chord polygon is shown in figure 1.

Therefore, we can now express the set of major and minor triads as $\langle n,n+4mod12,n+7mod12\rangle$ and $\langle n,n+8mod12,n+5mod12\rangle$ respectively for each n where $0\leq n\leq 11$ (note that $12=0_{mod12}$) on the musical clock. These ordered sets are called *pitch-class segments* in the music literature. The collection of the major and minor triads formed a set, called set of *consonant triads* because of their smooth sound. There is no restriction in the ordering of the elements due to the simultaneously sounding notes [13].

The above expressions for the major and minor triads reflect the component-wise action of the t/f – group. The expressions reveal that when the transposition t^n is applied to an entry when n=1, gives the entry immediately below it. For example, $t\langle 9,1,4\rangle = \langle t(9),t(1),t(4)\rangle = \langle 10,2,5\rangle$, and when n=2 gives the next second entry below it, i.e. $t^2\langle 9,1,4\rangle = \langle t^2(9),t^2(1),t^2(4)\rangle = \langle 11,3,6\rangle$, and so on. And generally, taking the first entry to be zero, i.e. the C-major triad, the nth entry in the first column is

$t^n\langle 0,4,7\rangle = \langle t^n(0),t^n(4),t^n(7)\rangle$ and the nth entry in the second column is

$t^n f\langle 0,4,7\rangle = \langle t^n f(0),t^n f(4),t^n f(7)\rangle$.

Hence, we conclude that for any consonants triads X and Y, there is a unique element h of the t/f – group such that hX = Y showing that t/f – group is transitive. This can be verified using the above expressions. The uniqueness of the element h is established using the orbit-stabilizer theorem proved in section 2.1.

VI. SUMMARY

The basic properties of the dicyclic group Q_{12} of order 24 that are concerned with music are discussed. The symmetry group Q_{12} is partitioned into two disjoint sets H and S consisting of transpositions and inversions respectively. The set H formed a cyclic group while the set S does not form a group. Likewise, (Z_{12}, -) does not form a group.

The orbits and stabilizer of the two disjoint subsets of Q_{12} are also analyzed where the stabilizer is found to be the trivial subgroup {1}, consisting of the identity transformation. From these results, we established the authenticity of the orbit-stabilizer theorem.

We also discussed the Conjugacy classes of Q_{12} and a pattern is derived for each Conjugacy class. And it is shown that the Conjugacy class is an equivalence relation. The centre of the group Q_{12} is a doublet $\{1, t^2\}$, a normal subgroup of Q_{12}.

Finally, the idea of transposition and inversion is discussed in details from the two perspectives, i.e. in mathematical terms and musical terms as well. These transpositions and inversions are the major connectors of algebra and music theory which are also used in the illustration of minor and major triads.

A. CONCLUSION

At this point of our discussion, we conclude that the Dicyclic group Q_{12} of order 24 and the set of integer modulo 12 play a vital role in the study of music, where the pitch classes can be translated in to integers modulo 12.

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