The Split Decompositions Of Finite Separable Metacyclic 2-Group

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Abstract: Given a finite separable metacyclic 2-group $G$, it is always possible to express $G$ as a semidirect product of a cyclic group with another cyclic group. In this paper, we implement the use of Group Application Package (GAP) Software to determine the split decompositions of a finite separable metacyclic 2-group up to isomorphism, where the dihedral group $D_{16}$ of order $2^4$ and its presentations was derived and shown to be separable. The finite groups were generated and expressed as the semidirect product of cyclic subgroups.

Keywords: Complement, Metacyclic group, Dihedral group; Separable, Semidirect product, Split decompositions

I. INTRODUCTION

A subgroup $N$ of a group $G$ is complemented in $G$ if there exists another subgroup $H$ of $G$ such that $G = NH$ and whenever $x \in N$ and $x \in H$, then $x = 1$, the identity element of $G$. i.e. $N \cap H = \{1\}$. If in addition, $N \triangleleft G$, then $G$ is said to split over $N$ and is written as $G = [N]H$. In this case, we say that $G$ is the semidirect product of its subgroups $N$ and $H$. If further, $G = [N]H = [N_1]H_1$, with $N \cong N_1$ and $H \cong H_1$, then the two split decompositions $[N]H$ and $[N_1]H_1$ are said to be isomorphic. A group is said to be separable if it splits over a nontrivial proper normal subgroup and inseparable otherwise.

Any metacyclic $p$-group can be presented by the relation $G = \langle x, y \mid x^{p^i} = 1, y^{p^j} = x^{t}, x^y = x^i \rangle$. If $G$ is separable, then by the same result, we can make $t$ to be 0 so that $G = \langle x, y \mid x^{p^i} = 1, y^{p^j} = x^{i} \rangle$. In this case, $G = \langle \langle x \rangle \rangle$ and in our result, we have shown that the separable metacyclic $p$-groups with $p$ odd have split decompositions isomorphic to $G$. However, this is not the case for all metacyclic 2-groups, particularly for Dihedral groups $D_n$ of order $2n$. For instance, consider the dihedral group $D_4 = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = 1, \alpha \beta = \beta \alpha \rangle$ of order 8. While $D_4 = \langle \langle \alpha \rangle \rangle \langle \beta \rangle$, it has a non-isomorphic decomposition, where $D_4 = \langle \langle \alpha \rangle \rangle \langle \beta \rangle \langle \alpha \beta \rangle$, with $\langle \alpha \beta \rangle$ isomorphic to the Klein 4-group. For general presentation and more information, see [2].

This paper is a review of the work of Kirtland, [10]. Our aim is to study the finite separable metacyclic 2-group, the necessary and sufficient conditions under which a separable metacyclic 2-group has all its split decompositions isomorphic as discussed by Kirtland and then implement the use of GAP to determine its splits decompositions. The concept of metacyclic 2-groups and metacyclic $p$-groups in general, have been extensively studied by many authors. For more information and results, see the work of Brandle and Verardi [3], Lie dhal [4], King [1], Beuerle [5], Hempel [6], and Sire [7]. In particular, separable metacyclic groups have been studied by Sim [8] and Jackson [9]. However, the simple condition for which a separable metacyclic 2-group has all its split decompositions isomorphic has not been directly addressed by these authors.

We used standard notations and for a $p$-group $G$, the subgroup $\Omega_i$ of $G$ is given by $\Omega_i(G) = \langle g \mid g^{p^i} = 1 \rangle$. The center of a group $G$ is denoted by $Z(G)$ and $\Phi(G)$ will denote its Frattini subgroup. Our research is limited to finite groups.

We shall see from the following theorem that if $G$ is a metacyclic $p$-group for $p$ an odd prime, then all its split decompositions are isomorphic.
THEOREM 1.1: Given a separable metacyclic $p$-group $G$ for $p$ an odd prime, then all its split decompositions are isomorphic [10].

**Proof.** By [1], we have $G = \langle x, y \mid x^{p^n} = y^{p^n} = 1, x^y = x^r \rangle$ and so, $G = \langle [x,y] \rangle$. Let $G = [N]H$ be another split decomposition for $G$. If $G$ is Abelian, then $G = \langle x \rangle \times \langle y \rangle = N \times H$ and without any loss of generality, we get by [10], that $N \cong \langle x \rangle$ and $H \cong \langle y \rangle$.

Assume $G$ is not Abelian. Then by [12], we have $|\Omega_2(G)| = p^2$ and since both $\Omega_2(N)$ and $\Omega_2(H)$ are nontrivial and both contained in $\Omega_2(G)$, it follows that $|\Omega_2(N)| = |\Omega_2(H)| = p$. But $p > 2$. Hence $N$ and $H$ are cyclic by [5] and by [7], $N \cong \langle x \rangle$ and $H \cong \langle y \rangle$.*

II. SEPARABLE METACYCLIC 2-GROUPS; $M = 1$

In this section, split decompositions of metacyclic 2-groups $G$ of the form $G = \langle x, y \mid x^{2^n} = y^2 = 1, x^y = x^r \rangle$ are investigated. Kirtland [10] in his paper also investigated the split decompositions of metacyclic 2-group. The novelty in this paper is to show how to investigate the split decompositions of a finite metacyclic 2-group using GAP Software and to determine if the split decompositions are isomorphic. If $G$ is abelian, then obviously, all its split decompositions are isomorphic by [11]. Otherwise, if $n = 2$, then $G$ is the dihedral group $D_8$, and all of its split decompositions are isomorphic [10]. The case $n \geq 3$ shall be considered in the next section where $G$ is non-Abelian.

We shall now follow the following theorem as stated by Kirtland [10], and then implement it in GAP by generating a Dihedral group $D_{16}$ of order 32 and its presentations.

**THEOREM 2.1:** Let $G$ be any non-Abelian metacyclic 2-group with presentation $G = \langle x, y \mid x^{2^n} = y^2 = 1, x^y = x^r \rangle$ with $n \geq 3$. Then the split decompositions of $G$ are isomorphic if and only if $r = 2^{n-1} + 1$.

**Proof:** Assume that all split decompositions of $G$ are isomorphic. Now consider the normal subgroup $N = \langle x^2, y \rangle$ of $G$. Since $N$ is complemented in $G$, we have $G = \langle [x^2, y] \rangle \langle x^2, y \rangle \cong \langle x, y \rangle$ and $\langle x^2, y \rangle$ is Abelian. Thus $x = (x^2)^y = (x^2)^r = x^2$ and $2r = 2$ (mod 2$^n$). Hence, $r = 1$ (mod 2$^{n-1}$) or $r = 2^{n-1} + 1$.

Conversely, suppose $r = 2^{n-1} + 1$ and that $G = [N]H$. Then obviously, $x = (x^2)^y = (x^2)^r = x^{2^r} = x^{2^2}$ and $\langle x \rangle = Z(G)$. Furthermore, $N \cap Z(G) \neq \{1\}$. This implies that $\langle x^{2^{n-1}} \rangle \subseteq N$ and $x^{2^{n-1}} \in H$. If in addition, $[x, y] = x^{-1}y^{-1}x^{-1}r^{-1} = x^{-1}y^{-1}x^{-1}r^{-1}$, then $G' = \langle x^{2^{n-1}} \rangle$.

Next, let $G' = G/(x^{2^{n-1}}) \cong Z_{2^{n-1}} \times Z_2$. Then consequently, $G = N/(x^{2^{n-1}}) \times H/(x^{2^{n-1}}) / (x^{2^{n-1}}) = \overline{N} \times \overline{H}$ with $\overline{N} \cong \{1\}$ and $\overline{H} \cong H$. Hence, we have by [11], that $\overline{N} \cong Z_{2^{n-1}}$ and $\overline{H} \cong H \cong Z_2$ or $\overline{N} \cong Z_2$ and $\overline{H} \cong H \cong Z_{2^{n-1}}$.

Finally, consider the case $\overline{N} \cong Z_2$ and $\overline{H} \cong H \cong Z_{2^{n-1}}$. Then we have $|N| = 4$ and $|H|/\phi(|N|) \leq |\text{Aut}(N)| = 2$. But if $H$ is cyclic, then we have $C_p(N) \leq H \cap Z(G) = \{1\}$ and $|H| \leq 2$. Thus $|G| = 8$ and $n = 2$. But this contradicts the fact that $n \geq 3$. Consequently, $\overline{N} \cong Z_{2^{n-1}}$ and $\overline{H} \cong H \cong Z_2 \cong \langle b \rangle$. Hence, $N$ is a maximal subgroup of $G$.

Now since $G/(x^2) = Z_2 \times Z_2$ and $\Phi(G) = \langle x^2 \rangle$, the only possible maximal subgroups of $G$ are $\langle x \rangle$, $\langle y \rangle$, and $\langle x^2, y \rangle$. But $(xy)^2 = xx^y = x^{r+1} = x^{2^{n-1}+2}$ and $|x^{2^{n-1}+2} = 2^{n-1}$.

Hence, $|xy| = 2^n$ and $\langle x^2, y \rangle \cong \langle xy \rangle \cong \langle x \rangle$. Now consider the subgroup $\langle x^2 \rangle$. Since $Z(G) = \langle x^2 \rangle$, $|\Omega_2(G)| = 4$ and $\Omega_2(G) = \langle x^{2^{n-1}} \rangle$, we have $\Omega_2(G) \cong \langle x^2, y \rangle$. Furthermore, $\Omega_1(H) \subseteq \Omega_1(G) \subseteq \langle x^2, y \rangle$. Therefore if $N = \langle x^2 \rangle$, then $N \cap H \neq \{1\}$, a contradiction. Hence, $N \cong \langle x \rangle$, $H \cong \langle y \rangle$ and all split decompositions are isomorphic.*

**REMARK 2.2:** Consider the dihedral group $D_{16}$, of order $2^4$ with the presentation $D_{16} = \langle \alpha, \beta \mid \alpha^{2^4} = \beta^2 = 1, \alpha^\beta = \alpha^r \rangle$.

We obtained the following results from GAP.

gap> G:= DihedralGroup(IsPermGroup,32);
gap> Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16), (2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10) ]);
gap> r:=(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16);
gap> N:= Subgroup(G, [r]);
gap> f:= (2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10);;
gap> H:= Subgroup(G, [f]);
gap> Size(N); Size(H);
gap> Size(D); 16
2
gap> IsCyclic(N); true
gap> IsCyclic(H); true
gap> IsNormal(G, N); true
gap> IsNormal(G, H); false
gap> D:= DirectProduct(N, H);
gap> Size(D); 32
gap> Center(G);
gap> Group([ (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) ]); gap> s:=(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16);
gap> Z:= Subgroup(G, [s]);
gap> Size(Z); 2
gap> IsNormal(G, Z); true
gap> IsCyclic(Z);
true
gap> quit

Here, the subgroup $N$ of order 16 consists of all the rotations in $D_{16}$ as follows:

If we let $\alpha = \langle 1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16 \rangle$, then $N$ is given by the following presentation:

$$N = \{ \alpha^n : 1 \leq n \leq 16, \text{ where } \alpha^{16} = \alpha \}.$$  

The cyclic subgroup $N$ of index 2 in $D_{16}$ is normal as shown above. It is also true by Lagrange’s theorem since $|D_{16}| = |N| = 2$. Then the only nontrivial elements $\{ \alpha^n \}$ of $G$. Hence, $N = \{ \alpha^n \}$, and $\alpha$ is odd. Then $G$ must all have its split decompositions isomorphic. We therefore implement the result using GAP Software by constructing some finite groups whose subgroups are cyclic.

**THEOREM 3.1:** If $G$ is a separable metacyclic 2-group of type $G = \langle x, y \mid x^{2m} = y^2 = 1, x^r = x' \rangle$ with $m \geq 2$, then all its split decompositions are isomorphic [10].

**PROOF.** If the group $G$ is Abelian, then the result follows directly from [11]. Now, supposed $G$ is not Abelian. Consequently, we shall have $r \geq 3$ and $n \geq 2$. We now consider three cases.

**CASE 1:** Let $a \in G$ where $a = x^n y^{2m-1} \alpha$ with $\alpha$ odd and $o(a) \neq 2$.

Now suppose $o(a) \neq 2$, then $a^2 = (x^n y^{2m-1})^2 = x^{2n} y^{2m-2} = 1$. This implies that $a + a \alpha = 0$ (mod $2^m$) or $\alpha (1 + r^{2m-1}) = k 2^n$ where $k$ is a positive integer. The fact that $\alpha$ is odd and $a \alpha^k$ implies that $\alpha k$. Thus $1 + r^{2m-1} = (k / \alpha) 2^n$ or $r^{2m-1} = (k / \alpha) 2^n - 1$. Hence, $x^{2m-1} = x^{2m-1} = x^{k / (k / \alpha) 2^n - 1} = x^{-1}$. Moreover, since $m \geq 2$, we have $x^{2m-1} = x^s$ where $1 \leq s \leq 2^m - 1$ and $s$ is odd. This implies that $s^2 = -1$ (mod $2^m$), a contradiction. Hence, $o(a) \neq 2$.

**CASE 2:** Let $\Omega_1(G) = \{ 1, x^{2m-1}, y^{2m-1}, x^{2m-1} y^{2m-1} \}$.

Now if $a = x^n y^m$ is a nontrivial element of $G$ such that $o(a) \neq 2$ and $a = 0$ or $b = 0$, then $a = y^{2m-1}$ or $a = x^{2m-1}$ respectively. If $a \neq 0$ and $b \neq 0$, then $\beta = 2^m$ and $a = x^n y^{2m-1}$.

Suppose $n = 2$. Then by case 1, $\alpha = 2 = 2^n$. But if $n \geq 3$ and since $\langle x \rangle$ acts on $\langle y \rangle$, there is a homomorphism $\Phi_i : \langle x \rangle \mapsto \text{Aut}(\langle x \rangle)$. Suppose that $\ker(\Phi_i) \neq 1$. Then $y^{2m-1} = \ker(\Phi_i)$ and $x$ and $y^{2m-1}$ commute. As a result, $1 = (x^n y^{2m-1})^2 = x^{2m} y^{2m} = x^{2m}$. This implies that $a = 2^n$ and $a = x^n y^{2m-1}$.

Next, suppose $\ker(\Phi_i) = 1$. Denote $\text{Aut}(\langle x \rangle)$ by $\{ \phi_1, \phi_2, ..., \phi_{2^n-1} \}$, where $\phi_i : x \mapsto x^i$, for $i = 1, 2, ..., 2^n - 1$. By [13], we get $\text{Aut}(\langle x \rangle) \cong L + K$, where $L$ is cyclic of order $2^{n-1}$ and generated by $\phi_2$, and $K$ is cyclic of order 2 generated by $\phi_{2^{n-1}}$. Then the only nontrivial elements of $\text{Aut}(\langle x \rangle)$ of order 2 are $\phi_{2^{n-1} + 1}, \phi_{2^{n-1} - 1}$, and $\phi_{2^{n-1}}$ and since $\langle y^{2m-1} \rangle = 2$, it follows that $\phi(y^{2m-1}) = \phi_{2^{n-1} + 1} \phi_{2^{n-1} - 1}$ or $\phi_{2^{n-1} - 1}$. Again if $m \geq 2$, then $\phi(y^{2m-2}) = \phi_{j}$ where $\phi(y^{2m-2}) = (\phi(y^{2m-2}))^2 = \phi_{j}^2$. Thus $j^2 = 2^{n-1} + 1$ (mod $2^m$), $2^{n-1} - 1$ (mod $2^m$), or $2^{n-1} - 1$ (mod $2^m$). But since $j^2 \neq -1$
Finally, given that \(\alpha^2 = 1\), then \((x^{\alpha} y^{x^{-1}})^2 = x^{\alpha^2} y^{x^{-1+1}} = x^{\alpha} y^{x^{-1+2}} = 1\) and since \(x^{2^n-1} = 1\), it follows that \(\alpha = 2^{n-1}\). Hence, \(\Omega_r(G) = \{1, x, x^2, y, x^2 y, x y^2, x y y^2\}\).

**CASE 3:** All split decompositions of \(G\) are isomorphic.

Supposed that \(G\) is given by the presentation \(G(n,m,r) = \langle x, y \mid x^{2^n-1} = 1, x^m = y^r, x^r = x^x \rangle\), and that \(G = [N]H\). Then \(G(n,m,r) / \langle x^{2^n-1} \rangle \cong G(n-1,m,r)\) and if \(n \geq 2\), then the result is true by induction on \(n\).

Again, since \([x, y] = x^{-1} y x^{-1} y^{-1}\), it implies that \(G^\prime = \langle x^{2^n-1} \rangle \leq \langle x \rangle\). Now let \(z = x^{2^n-1} \in \Omega_r(G^\prime) \cap Z(G)\). If \(z \notin N\), then \([G, N] \leq G^\prime / N \leq \langle x^{2^n-1} \rangle \cap N = \{1\} \). Thus \(N \leq Z(G)\) and so \(G = N \times H\). But this yields \([\Omega_r(G)] = 4\). Thus, we have \([\Omega_r(N)] = [\Omega_r(H)] = 2\). Consequently, \(N\) and \(H\) each have only one subgroup of order 2. But by \([12]\), \(N\) and \(H\) are either cyclic or a generalized quaternion group. In either case, both \(N\) and \(H\) are inseparable and thus by \([11]\), \(N \cong \langle x \rangle\) or \(H \cong \langle y \rangle\), or \(N \cong \langle y \rangle\) and \(H \cong \langle x \rangle\). In either case, \(H\) is cyclic, implying that \(G\) is abelian, a contradiction.

Hence, \(\langle z \rangle \leq N\) and \([\langle y \rangle / \langle z \rangle]\langle y \rangle \equiv G(n-1, m, r) \equiv G(z) \equiv [N / \langle z \rangle][H / \langle z \rangle] \langle z \rangle\). As a result, \(N / \langle z \rangle \cong \langle x \rangle / \langle z \rangle\) and \(H \cong \langle y \rangle / \langle z \rangle\) or \(N / \langle z \rangle \cong \langle y \rangle / \langle z \rangle\) and \(H \cong \langle x \rangle / \langle z \rangle\). In either case, \(H\) is cyclic and \(N\) is abelian. Also if \(N / \langle z \rangle \cong \langle y \rangle / \langle z \rangle\), then \(N\) has order 2 and is of exponent at least 2\(^{n-1}\). Thus \(N\) is isomorphic to \(Z_2^2\) or \(Z_{2^2} \times Z_2\). Finally, if \(N \cong Z_{2^{n-1}} \times Z_2\), then we have \([\Omega_r(N)] = 4\) and \([\Omega_r(H)] \geq 8\). This is a contradiction. Thus, \(N\) is cyclic and if \(N / \langle z \rangle \cong \langle y \rangle / \langle z \rangle\), then a similar argument yields that \(N\) must also be cyclic. Hence, both \(N\) and \(H\) are cyclic and by \([1]\), \(N \cong \langle x \rangle\) and \(H \cong \langle y \rangle\). *
function( x ) ... end

gap> N:= Subgroup(G, [f(a)]);
Group([ (1,7,13,3,9,15,5,11)(2,8,14,4,10,16,6,12),
(1,3,5,7,9,11,13,15)(2,4,6,8,10,12,14,16),
(1,5,9,13)(2,6,10,14)(3,7,11,15)(4,8,12,16),
(1,7,13,3,9,15,5,11)(2,8,14,4,10,16,6,12),
(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16),
(1,11,5,9,3,13,7)(2,12,6,14,10,8,16,4),
(1,13,9,5)(2,14,10,6)(3,15,11,7)(4,16,12,8),
(1,15,13,11,9,7,5,3)(2,16,14,12,10,8,6,4) ]
gap> Elements(N);
[ (), (1,3,5,7,9,11,13,15)(2,4,6,8,10,12,14,16),
(1,5,9,13)(2,6,10,14)(3,7,11,15)(4,8,12,16),
(1,7,13,3,9,15,5,11)(2,8,14,4,10,16,6,12),
(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16),
(1,11,5,9,3,13,7)(2,12,6,14,10,8,16,4),
(1,13,9,5)(2,14,10,6)(3,15,11,7)(4,16,12,8),
(1,15,13,11,9,7,5,3)(2,16,14,12,10,8,6,4) ]
gap> Size(N);
8
gap> IsNormal(G, N);
true
gap> f:= x^5;
function( x ) ... end
gap> K:= Subgroup(G, [f(a)]);
Group([ (1,6,11,16,5,10,15,4,9,14,3,8,13,2,7,12) ]
gap> Size(K);
16
gap> b:= (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16);
gap> M:= Subgroup(G, [b]);
Group([ (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) ]
gap> Elements(M);
[ (), (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16) ]
gap> Size(M);
2
gap> Size(G) = Size(M)*Size(N);
true

The subgroup \( N \) of \( G \) generated by the function \( f(x) = x^5 \) for an element \( x \in G \) is a normal subgroup while the function \( f(x) = x^3 \) generated the group \( K \cong G \). Hence, \( N \) is complemented in \( G \).

IV. CONCLUSION

In this paper, we have successfully shown that if \( G \) is a finite separable metacyclic 2-group with presentation \( G = \langle x, y \mid x^m = y^{2n} = 1, x^y = x^r \rangle, \ m \geq 1, \ n \geq 3 \), then all split decompositions of \( G \) are either isomorphic (if \( G \) is Abelian), or \( G \) is the Dihedral group of order \( 2n \). We therefore conclude that every finite group with the given presentation can be expressed as a semidirect product of a cyclic group with another cyclic group.

REFERENCES