Inequalities For Singular Values And Traces Of Quaternion Hermitian Matrices

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Abstract: *Let $A_1, A_2, \dots, A_m \in H_{n\times n}$ and $\alpha_1, \alpha_2, \dots, \alpha_m$ be positive real numbers. It is proved that if $\sum_{s=1}^{m} \alpha_s \geq 1$ and if the A_s are non negative definite triple complex matrices, then $|tr \prod_{s=1}^{m} A_s^{\alpha_s}| \leq \prod_{s=1}^{m} (trA_s)^{\alpha_s}$ and equality occurs if and only if (a) for $\sum_{s=1}^{m} \alpha_s = 1$, all A_s are scalar multiples of one another.*

(b) for $\sum_{s=1}^{m} \alpha_s = 1$, all A_s are scalar multiples of A_1 and are of rank 1. This result generalizes many classical inequalities and gives a multivariate version of the recent paper by Magnus (1987). The above inequality can be generalized further. Let $\sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C)$ be singular values of an $n \times n$ quaternion Hermitian matrices. Then for all $l = 1,2,\dots,n$ $\sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_s) \leq \sum_{t=1}^{l} \prod_{s=1}^{m} \sigma_t(A_s) \leq \prod_{s=1}^{m} \{\sum_{t=1}^{l} [\sigma_t(A_s)]^{1/\alpha_s}\}^{\alpha_s} \leq \sum_{s=1}^{m} \{\sum_{t=1}^{l} \alpha_s [\sigma_t(A_s)]^{1/\alpha_s}\}$

Keywords: *Hermitian Matrices, Quaternian Matrices, Singular value, Trace of Matrix, Triple Complex Matrices.*

I. INTRODUCTION

Throughout the paper, $H_{n\times n}$ will denote the $n \times n$ quaternion Hermitian matrices.

II. INEQUALITIES FOR SINGULAR VALUES

Let $C \in H_{n\times n}$. Then $\sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C)$ will denote the singular values of C, and $\sigma(C)$ will denote the column vector $(\sigma_t(C))_{t=1}^{n}$ in the n – dimensional Euclidean space $\mathbb{R}^n$; the $x_{[t]}'s$ will denote the rearrangement of the $x_t's$ with $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. Let $x = (x_t)$, $y = (y_t) \in \mathbb{R}^n$. Then x is said to be weakly majorized by y, if $\sum_{t=1}^{l} x_{[t]} \leq \sum_{t=1}^{l} y_{[t]}$ for all $l = 1,2,\dots,n$;

x is said to be majorized by y if $\sum_{t=1}^{l} x_t = \sum_{t=1}^{l} y_t$. By a result of Gelfand and Naimark (1950), [or p. 248 of Marshall and Olkin (1979)], we obtain

$$\sigma(A_1A_2 \dots A_m) < \sigma(A_1) * \sigma(A_2) * \dots * \sigma(A_m) \tag{2.1}$$

$[\because \sigma(A_1) * \sigma(A_2) * \dots * \sigma(A_m) = \sigma(A_1)\sigma(A_2)\dots\sigma(A_m)]$

Where each $A_t \in H_{n\times n}$ and $*$ is the pointwise product if we view $x \in \mathbb{R}^n$ as a function on {1,2,…,n}. By a result of Fan (1951)

$$\sigma(A_1 + A_2 + \cdots + A_m) < \sigma(A_1) + \sigma(A_2) + \cdots + \sigma(A_m) \tag{2.2}$$

THEOREM 1

Let $A_1, A_2, \dots, A_m \in H_{n\times n}, \alpha_1, \alpha_2, \dots, \alpha_m > 0$ with

$\sum_{s=1}^{m} \alpha_s = 1$ *and* $l \in \{1,2,\dots,n\}$. *Then*

$$\sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_s) \leq \sum_{t=1}^{l} \prod_{s=1}^{m} \sigma_t(A_s) \leq \prod_{s=1}^{m} \left\{\sum_{t=1}^{l} [\sigma_t(A_s)]^{\frac{1}{\alpha_s}}\right\}^{\alpha_s} \leq \sum_{s=1}^{m} \left\{\sum_{t=1}^{l} \alpha_s [\sigma_t(A_s)]^{\frac{1}{\alpha_s}}\right\}$$

PROOF

Since $A_s = A_{s0} + A_{s1}j + A_{s2}k$

$$\prod_{s=1}^{m} A_s = \prod_{s=1}^{m} (A_{s0} + A_{s1}j + A_{s2}k)$$

$$\prod_{s=1}^{m} A_s = \prod_{s=1}^{m} A_{s0} + \prod_{s=1}^{m} A_{s1}j + \prod_{s=1}^{m} A_{s2}k \ (\because A * B = AB = A_0B_0 + A_1B_1j + A_2B_2k)$$

$$\sigma_t(\prod_{s=1}^{m} A_s) = \sigma_t(\prod_{s=1}^{m} A_{s0} + \prod_{s=1}^{m} A_{s1}j + \prod_{s=1}^{m} A_{s2}k)$$

$$\leq \sigma_t(\prod_{s=1}^{m} A_{s0}) + \sigma_t(\prod_{s=1}^{m} A_{s1}j) + \sigma_t(\prod_{s=1}^{m} A_{s2}k)$$

$$[\because \sigma(A_1A_2 \dots A_m) < \sigma(A_1) * \sigma(A_2) * \dots * \sigma(A_m)]$$

Now,

$$\sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_{s0}) \leq \sum_{t=1}^{l}\prod_{s=1}^{m} \sigma_t(A_{s0}) \qquad (By\ theorem\ 1)$$

$$\leq \Pi_{s=1}^{m} \left\{ \Sigma_{t=1}^{l} [\sigma_t(A_{s0})]^{\frac{1}{\alpha_s}} \right\}^{\alpha_s}$$

$$\leq \sum_{s=1}^{m} \left\{ \sum_{t=1}^{l} \alpha_s [\sigma_t(A_{s0})^{\frac{1}{\alpha_s}} \right\}$$

$$Therefore, \sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_{s0}) \leq \sum_{t=1}^{l}\prod_{s=1}^{m} \sigma_t(A_{s0}) \leq \prod_{s=1}^{m} \left\{ \sum_{t=1}^{l} [\sigma_t(A_{s0})]^{\frac{1}{\alpha_s}} \right\}^{\alpha_s}$$

$$\leq \sum_{s=1}^{m} \left\{ \sum_{t=1}^{l} \alpha_s [\sigma_t(A_{s0})^{\frac{1}{\alpha_s}} \right\} \qquad (1)$$

Similarly,

$$\sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_{s1}j) \leq \sum_{t=1}^{l}\prod_{s=1}^{m} \sigma_t(A_{s1}j)$$

$$\leq \prod_{s=1}^{m} \left\{ \sum_{t=1}^{l} [\sigma_t(A_{s1}j)]^{\frac{1}{\alpha_s}} \right\}^{\alpha_s}$$

$$\leq \sum_{s=1}^{m} \left\{ \sum_{t=1}^{l} \alpha_s [\sigma_t(A_{s1}j)]^{\frac{1}{\alpha_s}} \right\}$$

$$Therefore, \sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_{s1}j) \leq \sum_{t=1}^{l}\prod_{s=1}^{m} \sigma_t(A_{s1}j) \leq \prod_{s=1}^{m} \left\{ \sum_{t=1}^{l} [\sigma_t(A_{s1}j)]^{\frac{1}{\alpha_s}} \right\}^{\alpha_s}$$

$$\leq \sum_{s=1}^{m} \left\{ \sum_{t=1}^{l} \alpha_s [\sigma_t(A_{s1}j)^{\frac{1}{\alpha_s}} \right\} \qquad (2)$$

Similarly,

$$\sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_{s2}k) \leq \sum_{t=1}^{l}\prod_{s=1}^{m} \sigma_t(A_{s2}k)$$

$$\leq \prod_{s=1}^{m} \left\{ \sum_{t=1}^{l} [\sigma_t(A_{s2}k)]^{\frac{1}{\alpha_s}} \right\}^{\alpha_s}$$

$$\leq \sum_{s=1}^{m} \left\{ \sum_{t=1}^{l} \alpha_s [\sigma_t(A_{s2}k)^{\frac{1}{\alpha_s}} \right\}$$

$$Therfore, \sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_{s2}k) \leq \sum_{t=1}^{l}\prod_{s=1}^{m} \sigma_t(A_{s2}k) \leq \prod_{s=1}^{m} \left\{ \sum_{t=1}^{l} [\sigma_t(A_{s2}k)]^{\frac{1}{\alpha_s}} \right\}^{\alpha_s}$$

$$\leq \sum_{s=1}^{m} \left\{ \sum_{t=1}^{l} \alpha_s [\sigma_t(A_{s2}k)^{\frac{1}{\alpha_s}} \right\} \qquad (3)$$

From (1), (2) and (3)

$$\sum_{t=1}^{l} \sigma_t(\prod_{s=1}^{m} A_s) \leq \sum_{t=1}^{l}\prod_{s=1}^{m} \sigma_t(A_s) \leq \prod_{s=1}^{m} \left\{ \sum_{t=1}^{l} [\sigma_t(A_s)]^{\frac{1}{\alpha_s}} \right\}^{\alpha_s}$$

$$\leq \sum_{s=1}^{m} \left\{ \sum_{t=1}^{l} \alpha_s [\sigma_t(A_s)^{\frac{1}{\alpha_s}} \right\}$$

The proof is completed.

THEOREM 2

Let $A_1, A_2, \dots, A_m \in H_{n\times n}, Where\ p > 1, and\ l \in \{1,2,\dots,n\}. Then$

$$\left\{ \sum_{t=1}^{l} \left[\sigma_t(\sum_{s=1}^{m} A_s) \right]^p \right\}^{1/p} \leq \left\{ \sum_{t=1}^{l} \left[\sum_{s=1}^{m} \sigma_t(A_s) \right]^p \right\}^{1/p} \leq \left\{ \sum_{s=1}^{m} \left[\sum_{t=1}^{l} \sigma_t(A_s) \right]^p \right\}^{1/p}$$

PROOF

Since $A = A_0 + A_1j + A_2k$

$$\sum_{s=1}^{m} A_s = \sum_{s=1}^{m} A_{s0} + \sum_{s=1}^{m} A_{s1}j + \sum_{s=1}^{m} A_{s2}k$$

$$\sigma_t(\sum_{S=1}^{m} A_s) = \sigma_t(\sum_{S=1}^{m} A_{s0} + \sum_{S=1}^{m} A_{s1}j + \sum_{S=1}^{m} A_{s2}k)$$

$$\leq \sigma_t(\sum_{S=1}^{m} A_{s0}) + \sigma_t(\sum_{S=1}^{m} A_{s1}j) + \sigma_t(\sum_{S=1}^{m} A_{s2}k)$$

$$\sum_{t=1}^{l} [\sigma_t(\sum_{S=1}^{m} A_s)] \leq \sum_{t=1}^{l} [\sigma_t(\sum_{S=1}^{m} A_{s0}) + \sigma_t(\sum_{S=1}^{m} A_{s1}j) + \sigma_t(\sum_{S=1}^{m} A_{s2}k)]$$

Now,

$$\left\{ \sum_{t=1}^{l} \left[\sigma_t(\sum_{s=1}^{m} A_{s0}) \right]^p \right\}^{1/p} \leq \left\{ \sum_{t=1}^{l} \left[\sum_{s=1}^{m} \sigma_t(A_{s0}) \right]^p \right\}^{1/p}$$

$$\leq \left\{ \sum_{s=1}^{m} \left[\sum_{t=1}^{l} \sigma_t(A_{s0}) \right]^p \right\}^{1/p} \qquad (by\ theorem\ 2)$$

Therefore,

$$\left\{ \sum_{t=1}^{l} \left[\sigma_t(\sum_{s=1}^{m} A_{s0}) \right]^p \right\}^{1/p} \leq \left\{ \sum_{t=1}^{l} \left[\sum_{s=1}^{m} \sigma_t(A_{s0}) \right]^p \right\}^{1/p} \leq \left\{ \sum_{s=1}^{m} \left[\sum_{t=1}^{l} \sigma_t(A_{s0}) \right]^p \right\}^{1/p} \qquad (1)$$

Similarly,

$$\left\{ \sum_{t=1}^{l} \left[\sigma_t(\sum_{s=1}^{m} A_{s1}j) \right]^p \right\}^{1/p} \leq \left\{ \sum_{t=1}^{l} \left[\sum_{s=1}^{m} \sigma_t(A_{s1}j) \right]^p \right\}^{1/p} \leq \left\{ \sum_{s=1}^{m} \left[\sum_{t=1}^{l} \sigma_t(A_{s1}j) \right]^p \right\}^{1/p} \qquad (2)$$

and

$$\left\{ \sum_{t=1}^{l} \left[\sigma_t(\sum_{s=1}^{m} A_{s2}k) \right]^p \right\}^{1/p} \leq \left\{ \sum_{t=1}^{l} \left[\sum_{s=1}^{m} \sigma_t(A_{s2}k) \right]^p \right\}^{1/p} \leq \left\{ \sum_{s=1}^{m} \left[\sum_{t=1}^{l} \sigma_t(A_{s2}k) \right]^p \right\}^{1/p} \qquad (3)$$

Now, By (1), (2) and (3)

$$\sum_{t=1}^{l}[\sigma_t(\sum_{s=1}^{m}A_s)] = \sum_{t=1}^{l}[\sigma_t(\sum_{s=1}^{m}A_{s0}) + \sigma_t(\sum_{S=1}^{m}A_{s1}j) + \sigma_t(\sum_{S=1}^{m}A_{s2}k)]$$

$$= \sum_{t=1}^{l}\sigma_t(\sum_{s=1}^{m}A_{s0}) + \sum_{t=1}^{l}\sigma_t(\sum_{S=1}^{m}A_{s1}j) + \sum_{t=1}^{l}\sigma_t(\sum_{S=1}^{m}A_{s2}k)]$$

$$\leq \sum_{t=1}^{l}\sum_{s=1}^{m}\sigma_t(A_{s0}) + \sum_{t=1}^{l}\sum_{s=1}^{m}\sigma_t(A_{s1}j) + \sum_{t=1}^{l}\sum_{s=1}^{m}\sigma_t(A_{s2}k)$$

$$\left\{\sum_{t=1}^{l}[\sigma_t(\sum_{s=1}^{m}A_s)]^p\right\}^{1/p} \leq \left\{\sum_{t=1}^{l}\sum_{s=1}^{m}[\sigma_t(A_{s0})]^p\right\}^{1/p} + \left\{\sum_{t=1}^{l}\sum_{s=1}^{m}[\sigma_t(A_{s1}j)]^p\right\}^{1/p}$$

$$+\left\{\sum_{t=1}^{l}\sum_{s=1}^{m}[\sigma_t(A_{s2}k)]^p\right\}^{1/p} \quad (by\ Theorem\ 2)$$

$$\leq \left\{\sum_{t=1}^{l}[\sigma_t(\sum_{s=1}^{m}\sigma_t(A_s)]^p\right\}^{1/p} \qquad (4)$$

$$\left\{\sum_{t=1}^{l}[\sigma_t(\sum_{s=1}^{m}A_s)]^p\right\}^{1/p} \leq \sum_{s=1}^{m}\left\{\sum_{t=1}^{l}[\sigma_t(\sum_{s=1}^{m}A_{s0})]^p\right\}^{1/p} + \sum_{s=1}^{m}\left\{\sum_{t=1}^{l}[\sigma_t\sum_{s=1}^{m}(A_{s1}j)]^p\right\}^{1/p}$$

$$+\sum_{s=1}^{m}\left\{\sum_{t=1}^{l}[\sigma_t\sum_{s=1}^{m}(A_{s2}k)]^p\right\}^{1/p}$$

$$\leq \sum_{s=1}^{m}\left\{\sum_{t=1}^{n}[\sigma_t\sum_{s=1}^{m}(A_s)]^p\right\}^{1/p} \qquad (5)$$

From (4) and (5), we get

$$\left\{\sum_{t=1}^{l}[\sigma_t\sum_{s=1}^{m}(A_s)]^p\right\}^{1/p} \leq \left\{\sum_{t=1}^{l}[\sum_{s=1}^{m}\sigma_t(A_s)]^p\right\}^{1/p} \leq \sum_{s=1}^{m}\left\{\sum_{t=1}^{l}[\sigma_t(A_s)]^p\right\}^{1/p}$$

The proof is completed.

III. INEQUALITIES FOR TRACES

THEOREM 3

Let $A_1, A_2, \dots, A_m$ be non zero non negative definite quaternion hermitian matrices in $H_{n\times n}$ and $\alpha_1, \alpha_2, \dots, \alpha_m > 0$.

- ✓ Suppose that $\sum_{t=1}^{m}\alpha_t = 1$. Then $\left|tr(\prod_{S=1}^{m}A_s^{\alpha_s})\right| \leq \prod_{s=1}^{m}(trA_s)^{\alpha_s}$ (1) and equality occurs if and only if all A_s are scalar multiples of A_1.
- ✓ Suppose that $\sum_{t=1}^{m}\alpha_t > 1$. Then (1) holds and equality occurs if and only if all A_s are scalar multiples of A_1 and $r(A_1) = 1$.

PROOF

- ✓ For $C = (c_{ts}) \in H_{n\times n}$

$$t_{ra}C = \sum_{t=1}^{n}c_{tt}$$

$$|t_{ra}C| = \left|\sum_{t=1}^{n}c_{tt}\right|$$

$$\leq \sum_{t=1}^{n}|c_{tt}|$$

$$\leq \sum_{t=1}^{n}\sigma_t(C)$$

So,

$$\left|tr(\prod_{s=1}^{m}A_s^{\alpha_s}\right| \leq \sum_{t=1}^{n}\sigma_t(\prod_{s=1}^{m}A_s^{\alpha_s}) \qquad (2)$$

By Theorem (1),

$$\sum_{t=1}^{l}\sigma_t(\prod_{s=1}^{m}A_{s0}^{\alpha_s}) \leq \prod_{s=1}^{m}\left\{\sum_{t=1}^{l}[\sigma_t(A_{s0}^{\alpha_s})]^{\frac{1}{\alpha_s}}\right\}^{\alpha_s} \qquad (3)$$

Since A_s is non negative definite,
$\sigma_t(A_{s0}^{\alpha_s}) = \lambda_t(A_{s0}^{\alpha_s}) = \lambda_t(A_{s0})^{\alpha_s}$, where $\lambda_t(A_{s0})$ is the t^{th} largest eigenvalue of A_{s0}.

Thus,

$$\prod_{s=1}^{m}\left\{\sum_{t=1}^{l}\{\sigma_t(A_{s0}^{\alpha_s})\}^{\frac{1}{\alpha_s}}\right\}^{\alpha_s} = \prod_{s=1}^{m}(trA_{s0})^{\alpha_s} \qquad (4)$$

From (2), (3), and (4) we obtain (1)

Let, s =1,2,…,m and we write

$$A_{s0} = P_{s0}\,D_{s0}\,P_{s0}^{*} \qquad (5)$$

Where P_{s0} is unitary and $D_{s0} = (\delta_{tl}\lambda_l(A_{s0}))$ is diagonal. Thus

$$\left|tr(A_{10}^{\alpha_1}\,A_{20}^{\alpha_2}\dots A_{m0}^{\alpha_m})\right| = tr(P_{m0}^{*}\,P_{10}\,D_{10}^{\alpha_1}\,P_{10}^{*}\,P_{20}D_{20}^{\alpha_2}\,P_{20}^{*}\dots P_{m-1_0}^{*}P_{m0}\,D_{m0}^{\alpha_m} \qquad (6)$$

By a complex version of Theorem 5 of kiers and Ten Berge (1989),

$$\left|tr\left(P_{m0}^{*}\,P_{10}\,D_{10}^{\alpha_1}\,P_{10}^{*}\,P_{20}D_{20}^{\alpha_2}\,P_{20}^{*}\dots P_{m-1_0}^{*}P_{m0}\,D_{m0}^{\alpha_m}\right)\right| \leq tr\,(D_{10}^{\alpha_1}\,D_{20}^{\alpha_2}\dots D_{m0}^{\alpha_m}) \qquad (7)$$

and equality occurs if and only if

$P_{m0}^{*}\,P_{10} = \pm N_{m0}\,M_{10}^{*}$,

$P_{s-1_0}^{*}\,P_{s0} = N_{s-1_0}^{*}\,M_s^{*}$, $s = 2,3,\dots,m$ $\quad(8)$

For some unitary matrices $N_s, M_s, and\ L_s$ satisfying

$N_sC = M_sC = L_sC$ $\qquad (9)$

Where,

$C=(Ir, o)', L_{s0}\,D_{s0}^{\alpha_s} = D_{s0}^{\alpha_s}\,L_{s0}, r = \min_{1\leq s\leq m} r(D_{s0}^{\alpha_s})$ $\quad (10)$

Since the product of two diagonal matrices is itself diagonal.

Here, $tr\left(A_1^{\alpha_1}A_2^{\alpha_2}\right) \leq (trA_1)^{\alpha_1}(trA_2)^{\alpha_2}\ gives$

$$tr(D_{10}^{\alpha_1}\,D_{20}^{\alpha_2}\dots D_{m0}^{\alpha_m}) \leq \prod_{s=1}^{m}(trD_{s0})^{\alpha_s} = \prod_{s=1}^{m}(trA_{s0})^{\alpha_s} \qquad (11)$$

and equality occurs if and only if for any $l = 2,3,\dots,m$

$D_{10} = a_{10}\,D_{10}$ for some $a_l > 0$ $\qquad (12)$

$So,$ $\quad r = r(D_{s0})$ for each S = 1,2, …, m. write

$$N_{10} = \begin{bmatrix} N_{11}^0 & N_{12}^0 \\ N_{21}^0 & N_{22}^0 \end{bmatrix}, L_{10} = \begin{bmatrix} L_{11}^0 & L_{12}^0 \\ L_{21}^0 & L_{22}^0 \end{bmatrix},$$

Where N_{11}^0 and L_{11}^0 are $r \times r$ matrices by (9) and (10)

$$N_{11}^0 = L_{11}^0, \qquad N_{21}^0 = L_{21}^0 \qquad (13)$$

Since L_{10} commutes with $D_{10}^{\alpha_1}$, it commutes with

$$D_{10} = \begin{bmatrix} D^0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (14)$$

Where D is non singular diagonal matrix. Thus

$$DL_{11}^0 = L_{11}D^0, L_{12}^0 = 0, L_{21}^0 = 0 \qquad (15)$$

By (15),

$$L_{10} = \begin{bmatrix} L_{11}^0 & 0 \\ 0 & L_{22}^0 \end{bmatrix} \qquad (16)$$

Since L_{10} is unitary.

$$L_{11}^{0^*} L_{11}^0 = I_r = L_{11}^{0^*}$$
$$L_{22}^{0^*} L_{22}^0 = I_{l-r} = L_{22}^* L_{22}^{0^*} \qquad (17)$$

Since N_{10} is unitary $N_{10} N_{10}^* = I_l$ and therefore

$$N_{11}^0 N_{11}^{0^*} + N_{12}^0 N_{12}^{0^*} = I_r \qquad (18)$$

By (13), (17) and (18), $N_{12}^0 N_{12}^{0^*} = 0, where$ $N_{12}^0 = 0, Thus$

$$N_{10} = \begin{bmatrix} N_{11}^0 & 0 \\ 0 & N_{22}^0 \end{bmatrix} = \begin{bmatrix} L_{11}^0 & 0 \\ 0 & N_{22}^0 \end{bmatrix} \qquad (19)$$

By (13), (15), and (19)

$$N_{s0} D_{10} = D_{10} N_{so}, \qquad M_{so} D_{10} = D_{10} M_{s0} \qquad (20)$$

By (5), (12) and (8)

$$A_{s0} = a_{s0} P_{s0} D_{10} P_{s0}^*$$

$$A_{s0} = a_{s0} P_{s0} D_{10} P_{s0}^*$$
$$= a_{s0} P_{10} P_{10}^* P_{20} P_{20}^* \dots P_{s0-1} P_{s0-1}^* P_{s0} D_{10} P_{s0}^* P_{s0-1} P_{s0-1}^* \dots P_{20} P_{20}^* P_{10} P_{10}^*$$
$$= a_{s0} P_{10} N_{10} M_{20}^* N_{20} M_{30}^* \dots N_{s0-1} M_{s0}^* D_{10} M_{s0} N_{s0-1}^* \dots M_{30} N_{20}^* M_{20} N_{10}^* P_{10}^*$$

so by (20),

$$A_{s0} = a_{s0} P_{10} D_{10} N_{10} M_{20}^* N_{20} M_{30}^* \dots N_{s0-1} M_{s0}^* M_{s0} N_{s0-1}^* \dots M_{30} N_{20}^* M_{20} N_{10}^* P_{10}^*$$

Since the N_{s0} and M_{s0} are unitary,

$$A_{s0} = a_{s0} P_{10} D_{10} P_{10}^* = a_{so} A_{10} \qquad (21)$$

Similarly,

$$A_{s1} = a_{s1} P_{11} D_{11} P_{11}^* = a_{s1} A_{11} \qquad (22)$$

and

$$A_{s2} = a_{s2} P_{12} D_{12} P_{12}^* = a_{s2} A_{12} \qquad (23)$$

Thus equality occurs in (1) only if (21), (22) and (23) holds. It is easy to prove that (21), (22) and (23) implies that equality occurs in (1).

(b) Let $\alpha = \sum_{s=1}^m \alpha_s$ then by (a)

$$|tr(\prod_{s=1}^m A_s^{\alpha^s})| = |tr[\prod_{s=1}^m (A_s^{\alpha})^{\frac{\alpha_s}{\alpha}}]| \le \prod_{s=1}^m (trA_s^{\alpha})^{\frac{\alpha_s}{\alpha}}$$

and equality occurs if and only if the A_s^{α} are scalar multiples of one another.

Note now that for any non zero non negative definite quaternion hermitian matrix A in $H_{n\times n}$.

$$trA^{\alpha} \le (trA)^{\alpha} \qquad (24)$$

and equality occurs if and only if A is of rank 1. So (1) holds, and equality occurs only if all A_s are scalar multiples of one another and are of rank 1.

The proof is completed.

THEOREM 4

Let $A_1, A_2, \dots, A_m$ be non zero non negative definite quaternion matrices in $H_{n\times n}$ and $\alpha_1, \alpha_2, \dots, \alpha_m$ be positive real numbers.

(a) Suppose that, $\sum_{s=1}^m \alpha_s = 1$. Then

$$\left| tr\left(\prod_{s=1}^m A_s^{\alpha_s}\right)\right| \le \prod_{s=1}^m (tr\, A_s)^{\alpha_s} \le \sum_{s=1}^m \alpha_s\, trA_s \qquad (1)$$

and equality in the right hand side occurs if and only if all trA_s are equal; hence equality occurs in the left hand side and in the right hand side if and only if all A_s are equal.

(b) Suppose that $\alpha = \sum_{s=1}^m \alpha_s > 1$. Then

$$\left| tr\left(\prod_{s=1}^m A_s^{\alpha_s}\right)\right| \le \prod_{s=1}^m (tr\, A_s)^{\alpha_s} \le \left(\sum_{s=1}^m \frac{\alpha_s}{\alpha}\, trA_s\right)^{\alpha} \qquad (2)$$

and equality in the right hand side occurs if and only if all $tr\, A_s$ are equal; hence equality occurs in the left hand side and in the right hand side if and only if all A_s are equal and are rank of 1.

NOTE THAT IN (2)

- ✓ if each $\alpha_s = 1$, then the inequality in the right hand side is nothing but the matrix version of the geometric - arithmetic mean inequality.
- ✓ if $\alpha < 1$, then the inequality in the right hand side holds; but the inequality in the left hand side may not hold.

THEOREM 5

Let t = 1,2,…,p , where $A_{ts}, s = 1,2,\dots,m$ are triple representation non zero non negative definite hermitian matrices in $H_{n\times n}$, and $\alpha_1, \alpha_2, \dots, \alpha_m$ be positive number such that

$\sum_{s=1}^m \alpha_s = 1$. Then

$$\sum_{t=1}^p |\, tr\, (\prod_{s=1}^m A_{ts})\,| \le \prod_{s=1}^m (\sum_{t=1}^p trA_{ts}^{\frac{1}{\alpha_s}})^{\alpha_s}$$

PROOF

Now,

$$\prod_{s=1}^m A_{ts} = \prod_{s=1}^m A_{ts_0} + \prod_{s=1}^m A_{ts_1} j + \prod_{s=1}^m A_{ts_2} k$$

$$tr\,(\prod_{s=1}^m A_{ts}) = tr\,[\prod_{s=1}^m A_{ts_0} + \prod_{s=1}^m A_{ts_1} j + \prod_{s=1}^m A_{ts_2} k]$$
$$\le tr\,(\prod_{s=1}^m A_{ts_0}) + tr\,(\prod_{s=1}^m A_{ts_1} j) + tr\,(\prod_{s=1}^m A_{ts_k} k)$$

$$|tr(\prod_{s=1}^{m} A_{ts})| \leq |tr(\prod_{s=1}^{m} A_{ts_0})| + |tr(\prod_{s=1}^{m} A_{ts_1} j)| + |tr(\prod_{s=1}^{m} A_{ts_2} k)|$$

$$\sum_{t=1}^{p} |tr(\prod_{s=1}^{m} A_{ts})| \leq \sum_{t=1}^{p} [tr(\prod_{s=1}^{m} A_{ts_0} + \prod_{s=1}^{m} A_{ts_1} j + \prod_{s=1}^{m} A_{ts_2} k)]$$

$$\leq \prod_{s=1}^{m} \{\sum_{t=1}^{p} [trA_{ts_0}] + \sum_{t=1}^{p} [trA_{ts_1}] j + \sum_{t=1}^{p} [trA_{ts_2}] k\}$$

$$\sum_{t=1}^{p} |tr(\prod_{s=1}^{m} A_{ts})| \leq \prod_{s=1}^{m} \{(\sum_{t=1}^{p} trA_{ts_0}^{\frac{1}{\alpha_s}})^{\alpha_s} + (\sum_{t=1}^{p} trA_{ts_1}^{\frac{1}{\alpha_s}})^{\alpha_s} j + (\sum_{t=1}^{p} trA_{ts_2}^{\frac{1}{\alpha_s}})^{\alpha_s} k\}$$

$$\leq \prod_{s=1}^{m} (\sum_{t=1}^{p} trA_{ts}^{\frac{1}{\alpha_s}})^{\alpha_s}$$

hence,

$$\sum_{t=1}^{p} |tr(\prod_{s=1}^{m} A_{ts})| \leq \prod_{s=1}^{m} (\sum_{t=1}^{p} trA_{ts}^{\frac{1}{\alpha_s}})^{\alpha_s}$$

The proof is completed.

THEOREM 6

Let A be a non zero non negative definite quaternian hermitian matrices in $H_{n\times n}$. $p_1, p_2, \dots, p_m$ be positive real numbers and $p = p_1$.

✓ Suppose that $\sum_{t=1}^{m} {}^{1}/_{p_t} = 1$.

Then for any non negative definite quaternion hermitian matrices $A_1, A_2, \dots, A_m$ with each

$$tr\ A_s^{p_s} = 1, |tr(A_s)| \leq [tr(A_s^p)]^{\frac{1}{p}} \qquad (1)$$

and equality occurs only if each $A_s^{p_s} = A_s^p / tr\ A_s^p$

✓ suppose that,

$\sum_{t=1}^{m} {}^{1}/_{p_t} > 1$ *and* $r(A) = 1$ *the conclusion of* (a) *stil hold.*

PROOF

✓ Let A be non zero non negative definite quaternion hermitian matrices. Then $A \in H_{n\times n}$.

Now,

$$A = A_0 + A_1 j + A_2 k$$
$$A_s = A_{s0} + A_{s1} j + A_{s2} k$$

$$tr(A_s) = tr(A_{s0} + A_{s1} j + A_{s2} k) \leq tr(A_{s0}) + tr(A_{s1}) j + tr(A_{s2}) k$$

$$|tr(A_s)| \leq |tr(A_{s0})| + |tr(A_{s1}) j| + |tr(A_{s2}) k|$$

$$\leq tr(A_{s0}^p)^{\frac{1}{p}} + tr(A_{s1}^p)^{\frac{1}{p}} j + tr(A_{s2}^p)^{\frac{1}{p}} k$$

$$\leq [tr(A_s^p)]^{\frac{1}{p}}$$

$$|tr(A_s)| \leq [tr(A_s^p)]^{\frac{1}{p}}$$

Hence the part (a).

✓ Suppose $|tr(A_s)| \leq [tr(A_s^p)]^{\frac{1}{p}}$ and equality occurs if and only if A is of rank 1. So (1) holds and equality occurs only if all A_s and are of rank 1.

Therefore r(A) = 1.

Hence the part (b).

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