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Quaternion Quasi-Normal Products Of Matrices

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Abstract: Here further properties of quasi-normal matrices are developed for quaternion matrices their relation, in a sense, to quaternion normal matrices is considered, and further results concerning quaternion quasi-normal products are obtained including for quaternion quasi-normal matrices.

Keywords: Quaternion unitary, quaternion hermitian, quaternion normal.

I. INTRODUCTION

A normal matrix $A=(a_{ij})$ with complex elements is a matrix such that $AA^{CT}=A^{CT}A$ where A^{CT} denotes the (complex) conjugate transpose of A. In an article by K. Morita[2] a quasi-normal matrix is defined to be a complex matrix A which is such that $AA^{CT}=A^{T}A^{C}$, where T denotes the transpose of A and A^{C} the matrix in which each element is replaced by its conjugate, and certain basic properties of such a matrix are developed there.

The main obstacle in the study of quaternion algebra is the non-commutative multiplication of quaternion. Many important conclusions over real and complex fields are different from ones over quaternion division algebra, such as determinant, the trace of matrix multiplication and solutions if quaternion equation. From the conclusions on quaternion division algebra, we find it to lack for general concepts, such as the definition of quaternion matrix determinant[5].

Recently, Wu in [7] used real representation methods to express quaternion matrices and established some new concepts over quaternion division algebra. From these definitions, we can see that they can convert quaternion division algebra problems into real algebra problems to reduce the complexity and abstraction which exits in all kinds of definitions given in [6].

It is possible if A is quaternion normal and B is quaternion quasi-normal that AB is quaternion quasi-normal[3]. For example, any quaternion quasi-normal matrix $C=HU=UH^{T}$ is such a product with $A=H$ and $B=U$, or if $C=HU=UH^{T}$ and $A=H$ then $AC=H^{2}U=U(H^{T})^{2}$. Therefore AC is quaternion quasi-normal. The following theorems clarify this matter.

THEOREM 1

If A is quaternion normal and B is quaternion quasi-normal then AB is quaternion quasi-normal if and only if $ABB^{CT}=BB^{CT}A$ and $B^{C}AA^{CT}=A^{T}A^{C}B^{C}$ (or $BA^{C}A^{T}=A^{CT}AB$).

PROOF

By the conditions, $(AB)(AB)^{CT}=ABB^{CT}A^{CT}=BB^{CT}AA^{CT}$ and $(AB)^{T}(AB)^{C}=B^{T}A^{T}A^{C}B^{C}=B^{T}B^{C}AA^{CT}$ which are equal.

Conversely, let AB is quaternion quasi-normal and let $UAU^{CT}=D=d_1I\oplus d_2I_2\oplus \ldots\ldots\oplus d_rI_r$ Where $d_p\bar{d}_p>d_q\bar{d}_q$, $p>q$. Let $UB^{T}U^{T}=B_1=(b_{pq})$. If $(AB)(AB)^{CT}=ABB^{CT}A^{CT}=AB^{T}B^{C}A^{CT}$

$=(AB)^{T}(AB)^{C}=B^{T}A^{T}A^{C}B^{C}=B^{T}A^{C}A^{T}B^{C}$. Therefore $(AB)(AB)^{CT}=(AB)^{T}(AB)^{C}=B^{T}A^{C}A^{T}B^{C}$, then

$(UAU^{CT})(UB^{T}U^{T}U^{C}B^{C}U^{CT})(UA^{CT}U^{CT})=$ $(UB^{T}U^{T})(U^{C}A^{C}U^{T}U^{C}A^{T}U^{T})(U^{C}B^{C}U^{CT})$

so that $DB_1B_1^{CT}D^{CT} = B_1D^CDB_1^{CT}$. Equating quaternion diagonal elements on each side of this relation

$\sum_{q=1}^{n} d_p\bar{d}_p b_{pq}\bar{b}_{pq} = \sum_{q=1}^{n} d_q\bar{d}_q b_{pq}\bar{b}_{pq}, \quad p = 1,2,\dots\dots n,$

or

$\sum_{q=1}^{n}(d_p\bar{d}_p - d_q\bar{d}_q)b_{pq}\bar{b}_{pq} = 0$

Let $d_1\bar{d}_1 = d_2\bar{d}_2 = \dots\dots = d_l\bar{d}_l > d_{l+1}\bar{d}_{l+1}$. Then $b_{pq} = 0$ for $p = 1,2,\dots\dots l$ and $q = l+1, l+2,\dots\dots n$. Since B_1 is quaternion quasi-normal, $\sum_{q=1}^{n} b_{pq}\bar{b}_{pq} = \sum_{q=1}^{n} b_{qp}\bar{b}_{qp}$ for $p = 1,2,\dots\dots n$.

On adding the first l of these equations and cancelling, $b_{pq} = 0$ for $p = l+1, l+2,\dots\dots n$ and $q = 1,2,\dots\dots n$. In this manner if $D = g_1D_1 \oplus g_2D_2 \oplus \dots\dots \oplus g_tD_t$ with $g_p > g_{p+1}$ and D_p is quaternion unitary, then $B_1 = C_1 \oplus C_2 \oplus \dots\dots \oplus C_t$ conformable to D. Since $g_pD_pD_p^{CT}g_pC_p^T = g_p^2C_p^T = C_p^Tg_p^2 = C_p^T r_p D_p D_p^{CT} r_p$, all p, $DD^{CT}B_1^T = B_1^TDD^{CT}$ and so $U^{CT}DD^{CT}UU^{CT}B_1^TU^C = U^{CT}B_1^TU^CU^TDD^{CT}U^C$ or $AA^{CT}B = BA^TA^C$ or $A^{CT}AB = BA^TA^C$ or $A^TA^CB^C = B^CAA^{CT}$.

Also $D(B_1B_1^{CT}D^{CT} = B_1D^CDB_1^{CT} = D^CDB_1^{CT} = D(D^CB_1B_1^{CT})$ so that $C_pC_p^{CT}(g_pD_p^C) = (g_pD_p^C)C_pC_p^{CT}$ for $p = 1,2,\dots\dots t$. (If $g_t = 0$, this is still true and D_t may be chosen to be the identity matrix.) Therefore $B_1B_1^{CT}D^{CT} = D^{CT}B_1B_1^{CT}$ and so $B^TB^CA^{CT}UB^TU^TU^CB^CU^{CT}UA^{CT}U^{CT} = UA^{CT}U^{CT}UB^TU^TU^CB_1^{CT}U^{CT}$

$= A^{CT}B^TB^C$ or $AB^TB^C = B^TB^CA$.

COROLLARY

Let A be quaternion normal, B is quaternion quasi-normal if AB is quaternion quasi-normal, then BA^C is quaternion quasi-normal and conversely.

PROOF

From the above, $UAU^{CT}UBU^T = DB_1^T$ is quaternion quasi-normal, and if $= D_gD_u$, D_g be real and D_u be quaternion unitary, then since $D_u^C = D_u^{CT}$, $D_u^C(DB_1^T)D_u^C = D_gB_1^TD_u^C = B_1^TD_gD_u^C = B_1^TD^C$ is quaternion quasi-normal as are $UBU^TU^CA^CU^T$ and BA^C. Reversing the steps proves the converse.

If A is quaternion normal, B is quaternion quasi-normal and BA^C is quaternion quasi-normal if and only if AB is quaternion quasi-normal if and only if $(B^TB^C)A = A(BB^{CT})$ and $(A^TA^C)B^C = B^C(AA^{CT})$. Therefore, if A is quaternion normal, B is quaternion quasi-normal and BA is quaternion quasi-normal if and only if $(B^TB^C)A^C = A^C(BB^{CT})$ and $(A^{CT}A)B^C = B^C(A^CA^T)$, that is replace A by A^C in the preceding, or $(B^{CT}B)A = A(B^CB^T) = A(B^{CT}B)$ and $(A^{CT}A)B^C = B^C(A^CA^T)$, thus exhibiting the fact that when AB is quaternion quasi-normal, BA is not necessarily so.

THEOREM 2

If $A = GW = WG$ is quaternion normal, and $B = SV = VS^T$ is quaternion quasi-normal (where G and S are quaternion hermitian and W and V are quaternion unitary) then AB is quaternion quasi-normal if and only if $GS = SG$, $GV = VG^T$ and $WS = SW$.

PROOF

If the three relations hold, then $AB = GWSV = GSWV$ on one hand, and $AB = WGSV = WSGV$

$= WSVG^T = WVS^TG^T = WV(GS)^T$ is quaternion quasi-normal[1], since GS is quaternion and WV is quaternion unitary.

Conversely, let

$A = U^{CT}DU = (U^{CT}D_gU)(U^{CT}D_uU) = GW$ and

$B = U^{CT}B_1^TU^C = (U^{CT}S_1U)(U^{CT}V_1U^C) = SV = VS^T$

where S_1 and V_1 are quaternion hermitian and quaternion unitary and direct sums conformable to B_1^T and D.

A direct check shows that $GS = SG$, $GV = VG^T$ also $WS = U^{CT}D_uS_1U = U^{CT}S_1D_uU = SW$ since $D_uB_1B_1^{CT} = B_1B_1^{CT}D_u$ implies $D_uK_1 = K_1D_u$.

NOTE

A sufficient condition for the simultaneous reduction of A and B is given by the following.

THEOREM 3

If A is quaternion normal, B is quaternion quasi-normal and $AB = BA^T$ then $WAW^{CT} = D$ and $WB^TW = F$, the quaternion normal form of Theorem 1[4], where W is a quaternion unitary matrix; also AB is quaternion quasi-normal.

PROOF

Let $UAU^{CT} = D$, quaternion diagonal, and $UBU^T = B_2$ which is quaternion quasi-normal. Then $AB = BA^T$ implies $DB_2 = UAU^{CT}UBU^T = UBU^TU^CA^TU^T = B_2D^T = B_2D$.

Let $D = c_1I_1 \oplus c_2I_2 \oplus \dots\dots \oplus c_mI_m$ where the c_p are quaternion and $c_p \neq c_l$ for $p \neq q$, and $B_2 = C_1 \oplus C_2 \oplus \dots\dots \oplus C_m$. Let V_p be quaternion unitary such that $V_pC_pV_p^T = F_p$ = the real quaternion normal form of Theorem 1[4], and let $V = V_1 \oplus V_2 \oplus \dots\dots \oplus V_m$. Then $VUAU^{CT}V^{CT} = D$, $VUBU^TV^T = F$ = a direct sum of the F_p. Also $AB = BA^T \Rightarrow B^TA^T = AB^T$ and so $ABB^{CT}A^{CT} = AB^TB^CA^{CT} = B^TA^TA^CB^C = (AB)^T(AB)^C$

It is also possible for the product of two quaternion normal matrices A and B to be quaternion quasi-normal. If $Q = HU = UH^T$ is quaternion quasi-normal and if $A = U$ and $B = H$ this is so or if $SV = VS^T$ is quaternion quasi-normal and if $A = US = SU$ is quaternion normal with S quaternion hermitian and V and U quaternion unitary, for $B = V$, $AB = (US)V = (UV)S^T$ is quaternion quasi-normal.

But if in the first example, U^2H is not quaternion normal, then HU is not quaternion quasi-normal, so that BA is not necessarily quaternion quasi-normal though AB is. When A alone is quaternion normal an analog of Theorem 2[4] can be obtained which states the following: If A is quaternion normal, then AB and AB^T are quaternion quasi-normal if and only if $ABB^{CT} = B^TB^CA$, $BB^{CT}A = AB^TB^C$, and $B^CAA^{CT} = A^TA^CB^C$. (The proof is not included here because of its similarity to that above.) When B is quaternion quasi-normal, two of these conditions merge into one in Theorem 1.

It is possible for the product of two quaternion quasi-normal matrices to be quaternion quasi-normal, but no such simple analogous necessary and sufficient conditions as exhibited above are available. This may be seen as follows. Two non-real quaternion commutative matrices $X = X^T$ and $Y = Y^T$ can form a quaternion quasi-normal (and non-real symmetric) matrix XY (Such that YX is also quaternion quasi-normal) which need not be quaternion normal. Then two symmetric matrices:

$$S = \begin{bmatrix} i & 1+i \\ 1+i & -i \end{bmatrix} \qquad T = \begin{bmatrix} 1+2i & 3-4i \\ 3-4i & -(1+2i) \end{bmatrix}$$

are such that $ST = Z$ is real, quaternion normal and quaternion quasi-normal (and not symmetric). Finally, if U and V are two quaternion unitary matrices of the same order, they can be chosen so UV is non-real quaternion, quaternion normal and quaternion quasi-normal. If $A = X \oplus S \oplus U$ and $B = Y \oplus T \oplus V$, $AB = XY + ST + UV$ where A and B are quaternion quasi-normal as in AB (but not symmetric). A simple inspection of these matrices shows that relations on the order of $(B^TB^C)A = A(BB^{CT}) = (BB^{CT})A$ and $(A^TA^C) = (AA^{CT})B^C$

$= B^C(AA^{CT})$ do not necessarily hold; these are sufficient, however, to guarantee that AB is quaternion quasi-normal (as direct verification from the definition will show).

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