International Journal of Innovative Research and Advanced Studies (IJIRAS)
Volume 4 Issue 7, July 2017

ISSN: 2394-4404

Local-Non Local Fractional Differential Equation In Benach Space

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***Abstract:** In this paper, we establish existence and uniqueness of solutions for a class of boundary value problem for fractional differential equations involving the Caputo fractional derivative in Banach Space.*

$${}^{c}D^{\alpha}y(t) = f(t, y(t)) + q(t, y(t))$$

For each $t \in J = [0, T]$, $1 < \alpha \le 2$ $y(0) = y_0$, $y(T) = y_T$

These results are obtained by using the fixed point technique.

I. INTRODUCTION

$${}^{c}D^{\alpha}y(t) = f(t, y(t)) + q(t, y(t)) \quad \text{for each}$$

$$t \in J = [0, T],\ 1 < \alpha \le 2 \quad (1.1)$$

$$y(0) = y_0, y(T) = y_T \quad (1.2)$$

Where ${}^{C}D^{\alpha}$ Caputo fractional derivative $f : [0, T] \times R \to R$ and $q : [0, T] \times R \to R$ both are continuous function and $y_T \in R$ differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and Engineering.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $y(0)$, $y'(0)$., etc. the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands.

For more details on the geometric and physical interpretation for fraction derivatives of both the Riemann-Liouville and Caputo types. In this project, we present existence and uniqueness results for the problem (1.1)-(1.2) involving Caputo's fractional derivatives. We give the results, one based on schaefer's fixed point theorem (Theorem 3.1) and another one based on Banach fixed point theorem (Theorem3.3) .Finally we present an example. These results can be considered as a contribution to this emerging field.

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper By C(J,R) we denote the Banach space of all continuous functions from J into R with the form,

$$\|y\|_{\infty} = \sup\{|y(t)| : t \in J\}$$

II. THE FRACTIONAL ORDER INTEGRAL

The fractional (arbitrary) order integral of the function $h \in L'([a,b], R_+)$ of order $\alpha \in R_+$ is defined by

$$I_a^{\alpha}h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

Where Γ is the gamma function when a=0, we write $I^{\alpha}h(t) = h(t) * \varphi_{\alpha}(t)$

Where $\varphi_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $t \succ 0$, and $\varphi_{\alpha}(t) = 0$ for $t \le 0$

and $\varphi_{\alpha} \to \delta(t)$

as $\alpha \to 0$, where δ is the delta function.

A. RIEMANN-LIOUVILLE FRACTIONAL ORDER DERIVATIVE

For a function h given on the interval [a,b] the α^{th} Riemann-Liouville fractional order derivative of h, is defined by

$$(D_{\alpha+}^{\alpha}h)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} h(s)\,ds$$

Hence, n=[∝]+1 and [∝] denote the integer part of ∝

B. CAPUTO FRACTIONAL ORDER DERIVATIVE

For a function h given on the interval [a,b], the Capto fractional order derivative of order ∝ of h, is defined by

$$\left({}^cD_{a+}^{\alpha}h\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\,ds$$ Where n=[∝]+1

C. STATEMENT AND FIRST CONSEQUENCES OF ARZELÀ–ASCOLI THEOREM

A sequence $\{f_n\}_{n\in N}$ of continuous function on an interval *I* = [*a*, *b*] is uniformly bounded if there is a number *M* such that

$$|f_n(x)| \le M$$

For every function f_n belonging to the sequence, and every *x*∈ [*a*, *b*]. The sequence is *equicontinuous* if, for every *ε*> 0, there exists *δ*> 0 such that

$$|f_n(x) - f_n(y)| < \varepsilon$$

Whenever $|x - y| < \delta$ for all functions f_n in the sequence. Succinctly, a sequence is equicontinuous if and only if all of its elements admit *the same* modulus of continuity. In simplest terms, the theorem can be stated as follows:

Consider a sequence of real-valued continuous functions $\{f_n\}_{n\in N}$ defined on a closed and bounded interval [*a*, *b*] of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence (f_{nk}) that converges uniformly.

The converse is also true, in the sense that if every subsequence of $\{f_n\}$ itself has a uniformly convergent subsequence, then $\{f_n\}$ is uniformly bounded and equicontinuous.

D. DEFINITION: EXISTENCE AND UNIQUENESS OF SOLUTION

Let us start by defining what we mean by a solution of the problem (1)-(2)

A function $y \in C^2([0,T],R)$ with its ∝-derivative exists on [0, T] is said to be a solution of (1)-(2) If y satisfies the equation, ${}^cD^{\alpha}y(t) = f(t,y(t)) + d(t,y(t))$ on J and conditions y(0)= y_0 and $y(T) = Y_T$

For the existence and uniqueness of solutions for the problem (1.1)-(1.2)

We need the following auxiliary lemmas:

LEMMA: 2.6

Let $\alpha > 0$, Then the fractional differential equation $D^{\alpha}h(t) = 0$ has solutions

$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots\ldots + c_{n-1}t^{n-1}$, $c_i \in R$, i=0, 1,………n-1, n=[∝]+1

LEMMA: 2.7

Let $\propto\, > 0$, then

$$I^{\alpha}D^{\alpha}\left(h(t)\right) = h(t) + c_0 + c_1 + c_2 + \ldots\ldots + c_{n-1}t^{n-1}$$

for some $c_i \in R$, i=0,1,2,……..n-1, n=[∝]+1

As a consequence of lemmas 3.2 and 3.3.we have the following result which is useful in what follows

LEMMA: 2.8

Let $1 < \alpha \le 2$ and Let $h:[0,T] \to R$ be continuous. A function y is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} h(s)\,ds - \left(\frac{t}{T}-1\right)y_0 + \frac{t}{T}y_T \to (3)$$

If and only if y is a solution of the fractional BVP.

$${}^cD^{\alpha}y(t) = h(t),\ t \in [0,T] \quad \to (4)$$

$$y(0) = y_0,\ y(T) = y_T \quad \to (5)$$

On the first result based on schaefer's fixed point theorem.

THEOREM: 2.9 (BANACH'S FIXED POINT THEOREM)

Let T be a contraction on a Banach space X. Then T has a unique fixed point.

THEOREM: 2.1.1 [SCHAEFER'S FIXED POINT THEOREM]

Assume that X is a Banach space and that $T : X \to X$ is a continuous compact mapping.Moreover assume that the set

$$\bigcup_{0\le\lambda\le1}\{x \in X : x = \lambda T(x)\}$$

Is bounded. Then T has a fixed point.

III. EXISTENCE OF SOLUTIONS

A. STATEMENT OF SCHAEFER'S FIXED POINT THEOREM

Assume that X is a Banach space and that $T : X \to X$ is a continuous compact mapping.Moreover assume that the set

$$\bigcup_{0\le\lambda\le1}\{x \in X : x = \lambda T(x)\}$$

Is bounded. Then T has a fixed point.

Using this statement for following existence theorem.

B. EXISTENCE SOLUTIONS OF THEOREM

Assume that,

[H1]: $f:[0,T]\times R\to R,\ q:[0,T]\times R\to R$ both are continuous function.

[H2]: There exists a constant k $\succ$ 0 such that

$$\left|f(s,y(s))\right|+\left|q(s,y(s))\right|\le k\left|y(s)\right|$$

Then the BVP (1.1)-(1.2) has at least one solution on [0, T].

PROOF

We shall use schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps.

STEP 1:

F is continuous

Let $\{y_n\}$ be a sequence such that,

$$y_n\to y \text{ in } C([0,T];R)$$

Then for each t∈ [0, T]

$$|F(y_n)(t)-F(y)(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\left[|f(s,y_n(s))+q(s,y_n(s))|-|f(s,y(s))+q(s,y(s))|\right]$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}ds\left[|f(s,y_n(s))+q(s,y_n(s))|-|f(s,y(s))+q(s,y(s))|\right]$$

$$\sup\{|F(y_n)(t)-F(y)(t)|\}\le\sup\left\{\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\left[(|f(s,y_n(s))|+|q(s,y_n(s))|)-(|f(s,y(s))|+|q(s,y(s))|)\right]\right.$$

$$\left.+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}ds\left[(|f(s,y_n(s))|+|q(s,y_n(s))|)-(|f(s,y(s))|+|q(s,y(s))|)\right]\right\}$$

$$\sup\{|F(y_n)(t)-F(y)(t)|\}\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\sup\left\{\left[(|f(s,y_n(s))|+|q(s,y_n(s))|)-(|f(s,y(s))|+|q(s,y(s))|)\right]\right\}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}ds\sup\left\{\left[(|f(s,y_n(s))|+|q(s,y_n(s))|)-(|f(s,y(s))|+|q(s,y(s))|)\right]\right\}$$

$$\sup\{|F(y_n)(t)-F(y)(t)|\}\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\sup\left\{\left[k|y_n(s)|-ky|y(s)|\right]\right\}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}ds\sup\left\{\left[k|y_n(s)|-ky|y(s)|\right]\right\}$$

$$\|F(y_n)-F(y)\|_\infty\le\frac{k.T^\alpha}{\Gamma(\alpha+1)}\|y_n-y\|_\infty+\frac{k.T^\alpha}{\Gamma(\alpha+1)}\|y_n-y\|_\infty$$

$$\|F(y_n)-F(y)\|_\infty\le\frac{2k.T^\alpha}{\Gamma(\alpha+1)}\|y_n-y\|_\infty$$

Since f and q are continuous functions, then we have,

$\|F(y_n)-F(y)\|_\infty$ as n→∞

STEP 2:

F maps bounded sets into bounded sets in $C([0,T];R)$

Indeed, it is enough to show that for any $\eta^*\ge 0$, There exists a positive constant l such that for each $y\in B_{\eta^*}=\{y\in C([0,T];R):\|y\|_\infty\le\eta^*\}$, we have $\|F(y)\|_\infty\le l$.

By[H1] and [H2] we have for each $t\in[0,T]$

$$|F(y)(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds+|y_T|$$

$$|F(y)(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[|f(s,y(s))+q(s,y(s))|\right]ds+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left[|f(s,y(s))+q(s,y(s))|\right]ds+|y_T|$$

$$|F(y)(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}k|y(s)|ds+\frac{1}{\Gamma(\alpha)}\int_0^T k|y(s)|ds+|y_T|$$

$$\le\frac{|y(s)|}{\Gamma(\alpha+1)}.T^\alpha+\frac{k|y(s)|}{\Gamma(\alpha+1)}.T^\alpha+|y_T|$$

$$\le\frac{2kT^\alpha}{\Gamma(\alpha+1)}|y(s)|+|y_T|$$

$$\sup\{|f(y)(t)|\}\le\sup\left\{\frac{2kT^\alpha}{\Gamma(\alpha+1)}|y(s)|+|y_T|\right\}$$

$$\le\frac{2kT^\alpha}{\Gamma(\alpha+1)}\sup\{|y(s)+|y_T||\}$$

$$\|f(y)\|_\infty\le\frac{2kT^\infty}{\Gamma(\alpha+1)}\|y\|_\infty+|y_T|$$

$$\le\frac{2kT^\alpha}{\Gamma(\alpha+1)}\eta^*+|y_T|\ \left(\because\eta^*\succ 0\right)$$

Thus , $\|F(y)\|_\infty\le l$ Where $l=\dfrac{2kT^\alpha}{\Gamma(\alpha+1)}\eta^*+\|y_T\|$

STEP 3:

F maps bounded sets into equicontinuous sets of $C([0,T];R)$.

Let $t_1,t_2\in[0,T],t_1\prec t_2,B\eta^*$ be a bounded set of $C([0,T];R)$ as in step 2, and Let $y\in B\eta^*$,

Then

$$|F(y)(t_1)-F(y)(t_2)|=\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[f(s,y(s))+d(s,y(s))\right]ds\right.$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[f(s,y(s))+d(s,y(s))\right]ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left[f(s,y(s))+d(s,y(s))\right]ds$$

$$\left.+\frac{(t_2-t_1)}{T}y_T\right|$$

$$|F(y)(t_1)-F(y)(t_2)|\le\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]|f(s,y(s))+q(s,y(s))|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[|f(s,y(s))|+|q(s,y(s))|\right]ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\left[f(s,y(s))+q(s,y(s))\right]ds$$

$$+\frac{(t_2-t_1)}{T}|y_T|.$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]|f(s,y(s))+q(s,y(s))|ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds$$

$$+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds+\frac{(t_2-t_1)}{T}|y_T|.$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]k|y(s)|ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}k|y(s)|ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}k|y(s)|ds$$

$$+\frac{(t_2-t_1)}{T}|y_T|.$$

$$\le\frac{k}{\Gamma(\alpha+1)}\left[(t_2-t_1)^\alpha+t_1^\infty-t_2^\infty\right]|y(s)|+\frac{k}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha|y(s)|+\frac{k(t_2-t_1)}{T\Gamma(\alpha+1)}.T^\alpha.|y(s)|$$

$$+\frac{t_2-t_1}{T}|y_T|$$

As $t_1\to t_2$, The right hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with

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International Journal of Innovative Research and Advanced Studies (IJIRAS)
Volume 4 Issue 7, July 2017
ISSN: 2394-4404

the Arzela-Ascoli theorem. We can conclude that $F:C([O,T],R)\to C([0,T],R)$ is completely continuous.

STEP 4:

A Priori bounds

Now, it remains to show that the set $\varepsilon=\{y\in C(J,R):Y=\lambda F(y)$ for some $0\prec\lambda\prec1\}$ is bounded.

Let $y\in\varepsilon$ Then $Y=\lambda F(y)$ for some $0\prec\lambda\prec1$ Thus for each $t\in y$ we have

$$|F(y)(t)|=\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,y(s))+q(s,y(s))]ds+\frac{\lambda t}{T\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}[f(s,y(s))+q(s,y(s))]ds$$

$$-\lambda\left(\frac{t}{T}-1\right)y_0+\lambda\frac{t}{T}y_T.$$

The implies by [H2] that for each $t\in J$ we have

$$|F(y)(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds+|y_T|$$

$$|F(y)(t)|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}dsk|y(s)|+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}dsk|y(s)|+|y_T|$$

$$\le\frac{T^\alpha}{\Gamma(\alpha+1)}k|y(s)|+\frac{T^\alpha}{\Gamma(\alpha+1)}k|y(s)|+|y_T|$$

$$\sup\{|F(y)(t)|\}\le\frac{2T^\alpha}{\Gamma(\alpha+1)}\sup\{|y(s)|+|y_T|\}$$

Then for every $t\in[0,T]$,we have,

$$\|F(y)\|_\infty\le\frac{2T^\alpha}{\Gamma(\alpha+1)}\|y\|_\infty+|y_T|$$

$$\le\frac{2T^\alpha}{\Gamma(\alpha+1)}\eta^*+|y_T|\ (\eta^*\succ0)$$

=R Where $R=\frac{2T^\alpha}{\Gamma(\alpha+1)}\eta^*+|y_T|$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem. We deduce that F has a fixed point which is a solution of the problem (1.1)-(1.2).

UNIQUENESS OF SOLUTIONS

C. STATEMENT OF BANACH FIXED POINT THEOREM

Let T be a contraction on a Banach space X. Then T has a unique fixed point.

Using this statement for following Uniqueness theorem

D. UNIQUENESS SOLUTIONS OF THEOREM

Assume that,

[H1]: $f:[0,T]\times R\to R,\ q:[0,T]\times R\to R$ both are continuous function.

[H2]: There exists a constant k ≻ 0 such that

$$|f(s,y(s))|+|q(s,y(s))|\le k|y(s)|$$

If $\frac{2kT^\alpha}{\Gamma(\alpha+1)}<1$ (6)

Then the BVP (1.1)-(1.2) has unique solution.

PROOF

Transform the problem (1.1)-(1.2) in to a fixed point problem consider the operator,

$$F:C([0,T],R)\to C([0,T],R)$$

Defined by

$$F(y)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,y(s))+q(s,y(s))]ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}[f(s,y(s))+q(s,y(s))]ds+\left(\frac{t}{T}-1\right)y_0+\frac{t}{T}y_T$$

Clearly, the fixed point of the operator F are solution of the (1)-(2).we shall use the Banach contraction principle to P.T, F has a fixed point. We shall show that F is a contraction.

Let $x,y\in C([0,T],R)$ Then for each t∈J we have,

$$|F(y)(t)-F(X(x)(t))|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|[f(s,y(s))+q(s,y(s))]\|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\|[f(s,y(s))+q(s,y(s))]\|ds+\left(\frac{t}{T}-1\right)y_0+\frac{t}{T}y_T$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|[f(s,x(s))+q(s,x(s))]\|ds-\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\|[f(s,x(s))+q(s,x(s))]\|ds$$

$$-\left(\frac{t}{T}-1\right)|y_0|-\frac{t}{T}|y_T|$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[|f(s,y(s))|+|q(s,y(s))|]ds+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}[|f(s,y(s))|+|q(s,y(s))|]ds$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[|f(s,x(s))|+|q(s,x(s))|]ds-\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}[|f(s,x(s))|+|q(s,x(s))|]ds$$

$$\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}dsk[|y(s)-x(s)|]+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}dsk[|y(s)-x(s)|]$$

$$\le\frac{1}{\Gamma(\alpha+1)}.T^\alpha.k[|y(s)-x(s)|]+\frac{1}{\Gamma(\alpha+1)}.T^\alpha.k[|y(s)-x(s)|]$$

$$\le\frac{2kT^\alpha}{\Gamma(\alpha+1)}|y(s)-x(s)|$$

$$\sup\{|F(y)(t)-F(x)(t)|\}\le\frac{2kT^\alpha}{\Gamma(\alpha+1)}\sup\{|y(s)-x(s)|\}$$

$$\|F(y)-F(x)\|_\infty\le\frac{2kT^\alpha}{\Gamma(\alpha+1)}\|y-x\|_\infty,\left(S=\frac{2kT^\alpha}{\Gamma(\alpha+1)}\right)$$

$$\|F(y)-F(x)\|_\infty\le S\|y-x\|_\infty$$

Consequently F is a contraction. As a consequence of Banach fixed point theorem. We deduce that F has a fixed point, which is a solution of a problem (1.1)-(1.2)

EXAMPLE

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem.

$${}^cD^\alpha y(t)=\frac{e^{-t}|y(t)|}{(9+e^t)(1+|y(t)|)}\quad t\in J=[0,1],\ 1\prec\alpha\le2\ \to(7)$$

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$y(0)=0 \; y(1)=0 \quad \to (8)$

Let $x, y \in [0,\infty)$ and $t \in J$ Then we have,

$$|F(t,x)-F(t,y)| = \frac{e^t}{(9+e^t)}\left|\frac{x}{1+x}-\frac{y}{1+y}\right|$$

$$= \frac{e^{-t}|x-y|}{(9+e^t)(1+x)(1+y)}$$

$$\le \frac{e^{-t}}{(9+e^t)}|x-y|$$

$$\le \frac{1}{10}|x-y|$$

Hence the condition [H2] holds with $k=\frac{1}{10}$

Hence [H2] is satisfied with T=1

Indeed

$$\frac{2kT^\alpha}{\Gamma(\alpha+1)} = \frac{2.\frac{1}{10}.(1)^\alpha}{\Gamma(\alpha+1)}$$

$$= \frac{1}{5\Gamma(\alpha+1)} \prec 1$$

$$\Gamma(\alpha+1) > \frac{1}{5}$$

$$\frac{2kT^\alpha}{\Gamma(\alpha+1)} < 1 <=> \Gamma(\alpha+1) > \frac{1}{5}$$

Which satisfied for each $\alpha \in (1,2]$

Then by uniqueness theorem. The problem (7)-(8) has a unique solutions on [0,1]

IV. NON-NOCAL PROBLEMS

Non-local fractional differential equation:

This section concerned with a generalization of the results presented is the previous section to nonlocal fractional differential equations. More precisely we shall present some existence and uniqueness results for the following nonlocal problem

$^cD^\alpha y(t)=f(t,y(t))+d(t,y(t))$ For each $t \in J=[0,T]$, $1<\alpha\le 2$ (9)

$$y(0)=y_0-g(y),\quad y(T)=y_T \qquad (10)$$

$g: C(J,R) \to R$ is continuous function. Nonlocal conditions were initiated by Byszewski $[28]$ when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski $[28,29]$, The nonlocal condition can be more useful than the standard initial conditions to describe some physical phenomena. For example, $g(y)$ may be given by

$$g(y)=\sum_{i=1}^{p} c_i y(\tau_i) \qquad (11)$$

Where $c_i, i=1,\ldots\ldots.,p,$ are given constants and $0<\tau_1<\ldots..<\tau_p\le T,$ to describe the diffusion phenomenon of small amount of gas in a transparent tube. In this case, (3) allows the additional measurements at t_i, $i=1,\ldots..p.$

Let us introduce the following set of conditions.

[H1]: $f:[0,T]\times R \to R,\; q:[0,T]\times R\to R$ both are continuous function.

[H2]: There exists a constant $k>0$ such that

$$|f(s,y(s))|+|q(s,y(s))|\le k|y(s)|$$

$[H3]$: There exist a constant $k^*>0$ such that,

$$|g(u)|\le k^*$$

$[H4]$: There exist a constant $k^{**}>0$ such that,

$\left|g(u)-g(\bar{u})\right|\le k^{**}|u-\bar{u}|$ For each $t\in T$ and all $u,\bar{u}\in C([0,T],R)$

A. EXISTENCE SOLUTIONS OF THEOREM

Assume that, assumptions $[H1]-[H2]-[H3]-[H4]$ hold. Then the BVP (9)-(10) has at least one solution on [0,T].

PROOF

We shall use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in several steps.

STEP 1

N is continuous

Let $\{y_n\}$ be a sequence such that,

$$y_n \to y \text{ in } C([0,T];R)$$

Then for each $t\in[0,T]$

$$|N(y_n)(t)-N(y)(t)|\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\big[|f(s,y_n(s))+q(s,y_n(s))|-|f(s,y(s))+q(s,y(s))|\big]$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds\big[|f(s,y_n(s))+q(s,y_n(s))|-|f(s,y(s))+q(s,y(s))|\big]+|g(y_n)-g(y)|$$

$$|N(y_n)(t)-N(y)(t)|\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\big[k|y_n|-k|y|\big]+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds\big[k|y_n|-k|y|\big]$$
$$+k^{**}|y_n-y|$$

$$|N(y_n)(t)-N(y)(t)|\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\big[k|y_n-y|\big]+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds\big[k|y_n-y|\big]$$
$$+k^{**}|y_n-y|$$

ISSN: 2394-4404

$$|N(y_n)(t)-N(y)(t)| \leq \frac{1}{\Gamma(\alpha)}.T^{\alpha}.\left[k|y_n-y|\right]+\frac{1}{\Gamma(\alpha)}.T^{\alpha}.\left[k|y_n-y|\right] +k^{**}|y_n-y|$$

$$\sup\{|N(y_n)(t)-N(y)(t)|\} \leq \frac{2kT^{\alpha}}{\Gamma(\alpha)}\sup\{|y_n-y|+k^{**}|y_n-y|\}$$

$$\left\|N(y_n)-N(y)\right\|_{\infty} \leq \frac{2kT^{\alpha}}{\Gamma(\alpha)}.\left\|y_n-y\right\|_{\infty}+k^{**}\left\|y_n-y\right\|_{\infty}$$

$$\left\|N(y_n)-N(y)\right\|_{\infty} \leq \left(\frac{2kT^{\alpha}}{\Gamma(\alpha)}+k^{**}\right)\left\|y_n-y\right\|_{\infty}$$

$$\left\|N(y_n)-N(y)\right\|_{\infty} \leq \left(\frac{2kT^{\alpha}}{\Gamma(\alpha)}+k^{**}\right)\left\|y_n-y\right\|_{\infty}$$

$$,\left(\frac{2kT^{\alpha}}{\Gamma(\alpha)}+k^{**}=S, k^{**}\succ 0\right)$$

$$\left\|N(y_n)-N(y)\right\|_{\infty} \leq S\left\|y_n-y\right\|_{\infty}$$

Since f and q and also g are continuous functions, then we have,

$$\left\|F(y_n)-F(y)\right\|_{\infty} \to 0 \text{ as } n\to\infty$$

STEP 2

N maps bounded sets into bounded sets in $C\left(\left[0,T\right];R\right)$

Indeed, it is enough to show that for any $\eta^{*} \geq 0$, there exists a positive constant l such that for each $y \in B_{\eta^{*}} = \left\{y \in C([0,T];R) : \|y\|_{\infty} \leq \eta^{*}\right\}$, we have $\left\|F(y)\right\|_{\infty} \leq l$. By

[H1] and [H2] we have for each $t \in \left[0,T\right]$

$$|N(y)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds +2|y_0-g(y)|+|y_T|$$

$$|N(y)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\left[|f(s,y(s))+q(s,y(s))|\right]+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds\left[|f(s,y(s))+q(s,y(s))|\right] +2|g(y)|+2|y_0|+|y_T|$$

$$|N(y)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}dsk|y(s)|+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}dsk|y(s)|+2k^{*}+2|y_0|+|y_T|$$

$$\leq \frac{k|y(s)|}{\Gamma(\alpha+1)}.T^{\alpha}+\frac{k|y(s)|}{\Gamma(\alpha+1)}.T^{\alpha}+2k^{*}+2|y_0|+|y_T|$$

$$\leq \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}|y(s)|+2k^{*}+2|y_0|+|y_T|$$

$$\sup\{|N(y)(t)|\} \leq \sup\left\{\frac{2kT^{\alpha}}{\Gamma(\alpha+1)}|y(s)|+2k^{*}+2|y_0|+|y_T|\right\}$$

$$\leq \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}\sup\{|y(s)|+2k^{*}+2|y_0|+|y_T|\}$$

$$\left\|N(y)\right\|_{\infty} \leq \frac{2kT^{\infty}}{\Gamma(\alpha+1)}\|y\|_{\infty}+k^{*}+2|y_0|+|y_T|$$

$$\leq \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}\eta^{*}+k^{*}+2|y_0|+|y_T| \left(\because \eta^{*}\succ 0\right),\left(k^{*}\succ 0\right)$$

Thus,

$\left\|N(y)\right\|_{\infty} \leq l$ Where

$$l = \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}\eta^{*}+k^{*}+2|y_0|+|y_T|$$

STEP 3

N maps bounded sets into equicontinuous sets of $C\left(\left[0,T\right];R\right)$.

Let $t_1, t_2 \in \left[0,T\right], t_1 \prec t_2, B\eta^{*}$ be a bounded set of $C\left(\left[0,T\right];R\right)$ as in step 2, and

Let $y \in B\eta^{*}$,

Then

$$|N(y)(t_1)-N(y)(t_2)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\left[f(s,y(s))+q(s,y(s))\right]ds\right.$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[f(s,y(s))+q(s,y(s))\right]ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[f(s,y(s))+q(s,y(s))\right]ds$$
$$\left.+\frac{(t_2-t_1)}{T}\left(y_0-g(y)\right)+\frac{(t_2-t_1)}{T}y_T\right|$$

$$|N(y)(t_1)-N(y)(t_2)| \leq \frac{1}{\Gamma(\alpha)}\int_0^{t}\left[(t_2)^{\alpha-1}-(t_1)^{\alpha-1}\right]\left[|f(s,y(s))+q(s,y(s))|\right]ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[|f(s,y(s))+q(s,y(s))|\right]ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[|f(s,y(s))+q(s,y(s))|\right]ds$$
$$+\frac{(t_2-t_1)}{T}(|y_0|+|g(y)|)+\frac{(t_2-t_1)}{T}|y_T|$$

$$|N(y)(t_1)-N(y)(t_2)| \leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]|f(s,y(s))+q(s,y(s))|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[|f(s,y(s))|+|q(s,y(s))|\right]ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds$$
$$+\frac{(t_2-t_1)}{T}\left(|y_0|+|g(y)|\right)+\frac{(t_2-t_1)}{T}|y_T|.$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]|f(s,y(s))+q(s,y(s))|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds$$
$$+\frac{(t_2-t_1)}{T}\left(|y_0|+|g(y)|\right)+\frac{(t_2-t_1)}{T}|y_T|.$$
$$\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]k|y(s)|ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}k|y(s)|ds+\frac{(t_2-t_1)}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}k|y(s)|ds\right.$$
$$+\frac{(t_2-t_1)}{T}|y_0|+\frac{(t_2-t_1)}{T}k^{*}+\frac{(t_2-t_1)}{T}|y_T|.$$

$$\leq \frac{k}{\Gamma(\alpha+1)}\left[(t_2-t_1)^{\alpha}+t_1^{\infty}-t_2^{\infty}\right]|y(s)|+\frac{k}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}|y(s)|+\frac{k(t_2-t_1)}{T\Gamma(\alpha+1)}T^{\alpha}|y(s)|$$
$$+\frac{(t_2-t_1)}{T}|y_0|+\frac{(t_2-t_1)}{T}k^{*}+\frac{(t_2-t_1)}{T}|y_T|$$

As $t_1 \to t_2$ The right hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem. We can conclude that $N : C\left(\left[O,T\right],R\right) \to C\left(\left[0,T\right],R\right)$ is completely continuous.

STEP 4

A Priori bounds

Now, it remains to show that the set

$$\varepsilon = \left\{ y \in C(J,R) : Y = \lambda F(y) \text{ for some } 0 \prec \lambda \prec 1 \right\}$$

is bounded.

Let $y \in \varepsilon$ Then $Y = \lambda F(y)$ for some $0 \prec \lambda \prec 1$ Thus for each $t \in y$ we have

$$|N(y)(t)| = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[|f(s,y(s))+q(s,y(s))|\right]ds + \frac{\lambda t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[|f(s,y(s))+q(s,y(s))|\right]ds$$
$$-\lambda\left(\frac{t}{T}-1\right)|y_0 - g(y)| + \lambda\frac{t}{T}|y_T|.$$

The implies by [H2] that for each $t \in J$ we have

$$|N(y)(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,y(s))+q(s,y(s))|ds$$
$$-\lambda\left(\frac{t}{T}-1\right)\left(|y_0| + |g(y)|\right) + \lambda\frac{t}{T}|y_T|$$

$$|N(y)(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} dsk|y(s)| + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} dsk|y(s)|$$
$$-\lambda\left(\frac{t}{T}-1\right)\left(|y_0| + k^*\right) + \lambda\frac{t}{T}|y_T|$$

$$|N(y)(t)| \le \frac{1}{\Gamma(\alpha+1)}.T^{\alpha}.k|y(s)| + \frac{1}{\Gamma(\alpha+1)}.T^{\alpha}.k|y(s)|$$
$$-\lambda\left(\frac{t}{T}-1\right)\left(|y_0| + k^*\right) + \lambda\frac{t}{T}|y_T|$$

$$\sup\left\{|N(y)(t)|\right\} \le \sup\left\{\frac{2kT^{\alpha}}{\Gamma(\alpha+1)}|y(s)| - \lambda\left(\frac{t}{T}-1\right)\left(|y_0| + k^*\right) + \lambda\frac{t}{T}|y_T|\right\}$$

$$\|F(y)\|_{\infty} \le \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\|y\|_{\infty} + |y_0| + k^* + |y_T|$$

Then for every $t \in [0,T]$,we have,

$$\|F(y)\|_{\infty} \le \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\|y\|_{\infty} + |y_0| + k^* + |y_T|$$

$$\le \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\eta^* + |y_0| + k^* + |y_T| \quad (\eta^* \succ 0),(k^* \succ 0)$$

=R Where $R = \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\eta^* + |y_0| + k^* + |y_T|$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem. We deduce that F has a fixed point which is a solution of the problem (9)-(10).

A. UNIQUENESS SOLUTION OF THEOREM

Assume that,

[H1]: $f : [0,T] \times R \to R,\ q : [0,T] \times R \to R$ both are continuous function.

[H2]: There exists a constant k > 0 such that

$$|f(s,y(s))| + |q(s,y(s))| \le k|y(s)|$$

If $\frac{2kT^{\alpha}}{\Gamma(\alpha+1)} + k^{**} < 1$ (12)

Then the BVP (9)-(10) has unique solution.

PROOF

Transform the problem (9)-(10) in to a fixed point problem consider the operator,

$$N : C([0,T],R) \to C([0,T],R)$$

Defined by

$$N(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s,y(s)) + q(s,y(s))\right]ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[f(s,y(s)) + q(s,y(s))\right]ds + \left(\frac{t}{T}-1\right)(y_0 - g(y)) + \frac{t}{T}y_T$$

Clearly, the fixed point of the operator F are solution of the (9)-(10).we shall use the Banach contraction principle to P.T, F has a fixed point.we shall show that F is a contraction.

Let $x, y \in C([0,T],R)$ Then for each t∈J we have,

$$|N(y)(t) - N((x)(t))| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|\left[f(s,y(s)) + q(s,y(s))\right]\right\|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left\|\left[f(s,y(s)) + q(s,y(s))\right]\right\|ds + \left(\frac{t}{T}-1\right)|y_0 - g(y)| + \frac{t}{T}|y_T|$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|\left[f(s,x(s)) + q(s,x(s))\right]\right\|ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left\|\left[f(s,x(s)) + q(s,x(s))\right]\right\|ds - \left(\frac{t}{T}-1\right)|y_0 - g(x)| - \frac{t}{T}|y_T|$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[|f(s,y(s))| + |q(s,y(s))|\right]ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[|f(s,y(s))| + |q(s,y(s))|\right]ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[|f(s,x(s))| + |q(s,x(s))|\right]ds - \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[|f(s,x(s))| + |q(s,x(s))|\right]ds$$
$$+|g(y) - g(x)|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}k|y(s)|ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}k|y(s)|ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}k|x(s)|ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}k|x(s)|ds + |g(y) - g(x)|$$

$$\le \frac{1}{\Gamma(\alpha+1)}.T^{\alpha}.k\left[|y(s) - x(s)|\right] + \frac{1}{\Gamma(\alpha+1)}.T^{\alpha}.k\left[|y(s) - x(s)|\right] + |g(y) - g(x)|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}dsk\left[|y(s) - x(s)|\right] + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}dsk\left[|y(s) - x(s)|\right] + |g(y) - g(x)|$$

$$\le \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}|y(s) - x(s)| + k^{**}|y - x|$$

$$\sup\left\{|N(y)(t) - N(x)(t)|\right\} \le \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}\sup\left\{|y(s) - x(s)| + k^{**}|y - x|\right\}$$

$$\|N(y) + N(x)\|_{\infty} \le \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}\|y - x\|_{\infty} + k^{**}\|y - x\|_{\infty}$$

$$\|N(y) + N(x)\|_{\infty} \le \left(\frac{2kT^{\alpha}}{\Gamma(\alpha+1)} + k^{**}\right)\|y - x\|_{\infty}, \left(S = \frac{2kT^{\alpha}}{\Gamma(\alpha+1)} + k^{**}\right)$$

$$\|N(y) - N(x)\|_{\infty} \le S\|y - x\|_{\infty}$$

Consequently F is a contraction. As a consequence of Banach fixed point theorem. We deduce that F has a fixed point, which is a solution of a problem (9)-(10)

NON-LOCAL EXAMPLE

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem.

$$ {}^{c}D^{\alpha}y(t)=\frac{e^{-t}|y(t)|}{(9+e^{t})(1+|y(t)|)}\quad t\in J=[0,1],\ 1\prec\alpha\leq 2\ \rightarrow(13) $$

$$ y(0)=\sum_{i=1}^{n}c_i y(t_i),\ y(1)=0\quad\rightarrow(14) $$

Where $0<t_1<t_2<\ldots\ldots<t_n<1 c_i,\ i=1,\ldots.n$ are given positive constants with $\sum_{n=1}^{n}c_i\prec\frac{4}{5}.$ set

$$ f(t,x)=\frac{e^{-t}x}{(9+e^{t})(1+x)},\ (t,x)\in J\times[0,\infty), $$

And $g(y)=\sum_{i=1}^{n}c_i y(t_i).$

Let $x,y\in[0,\infty)$ and $t\in J$ Then we have,

$$ |F(t,x)-F(t,y)|=\frac{e^{t}}{(9+e^{t})}\left|\frac{x}{1+x}-\frac{y}{1+y}\right| $$

$$ =\frac{e^{-t}|x-y|}{(9+e^{t})(1+x)(1+y)} $$

$$ \leq\frac{e^{-t}}{(9+e^{t})}|x-y| $$

$$ \leq\frac{1}{10}|x-y| $$

Hence the condition [H2] holds with $k=\frac{1}{10}$

Also we have,

$$ |g(x)-g(y)|\leq\sum_{i=1}^{n}c_i|x-y|. $$

Hence [H2] is satisfied with $k^{**}=\sum_{i=1}^{n}c_i$. We shall check that condition (12) with T=1.

Indeed

$$ \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}+k^{**}=\frac{2.\frac{1}{10}.(1)^{\alpha}}{\Gamma(\alpha+1)} $$

$$ =\frac{1}{5\Gamma(\alpha+1)}<1 $$

$$ \Gamma(\alpha+1)>\frac{1}{5} $$

$$ \frac{2kT^{\alpha}}{\Gamma(\alpha+1)}<1 <=> \Gamma(\alpha+1)>\frac{1}{5} $$

Which satisfied for each $\alpha\in(1,2]$

Then by uniqueness theorem. The problem (13)-(14) has unique solutions on [0, 1]

V. CONCLUSION

In this paper, I established existence and uniqueness of solutions for a class of boundary value problem for local and non-local fractional differential equations involving the Caputo fractional derivative in Banach Space. These results are obtained by using Banach fixed point theorem and schaefer's fixed point theorem.

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