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# A Latin Square Design With Three Restrictions On Randomization (3RR – Latin Square)

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***Abstract:** In this paper we present another variation of Latin square design called 3RR – Latin Square design. This design adds another restriction on randomization to the existing Latin square with two restrictions on randomization. Unlike Graeco Latin square design which adds randomization by using Greek letters treatments within $p^2$ cells along with the Latin letters treatment in such a manner that no combination of Greek and Latin letters is repeated, 3RR – Latin square design adds randomization by grouping experimental units into regions and applying every treatment in each region. The construction of the 3RR – Latin square design is illustrated by considering order 4, 6 and 9. A statistical model for the design is presented, parameters in the model estimated and the Analysis of Variance performed.*

***Keywords:** 3RR - Latin square, randomization, experimental error, ANOVA, blocking*



## I. INTRODUCTION

For purposes of error control or error reduction, the principle of blocking in designing experiment is of primary importance. In most situations blocking is one dimensional as in randomized block designs or two dimensional as in Latin square designs.

A Latin square design is an experimental design in which P treatments are arranged in a P x P array such that each treatment appears only once in a row and only once in a column. Thus, two restrictions are placed on randomization (rows and columns). This design has advantage over completely randomized design and randomized block design in the sense that experimental errors are reduced. Keptone (1952), Montgomery (1976)

For purposes of further reduction in error a number of associated designs have been developed. One of them is the Graeco – Latin square design. This design adds one more restriction on randomization and uses Greek letters within $P^2$ cells along with the Latin letters in such a manner that no combination of Greek – Latin letters are repeated.Bose et al (1960),

Another associated design is the Youden square which allows rectangular arrangements (Hicks, 1973). A very useful design for experimenters confronted with the problem that some of the experimental units are better than others or time causes a change in the units is the Cross Over design. Cochran and Cox (1957) show how a Cross Over design is a special case of the Latin square design.

In addition, there are systematic squares such as the Knut and Vik, which have been used by experimenters, but the experimental error is in question on these designs even if there are no interactions (Kempthorne, 1952; Fisher, 1966 and Yates, 1937).

The Sudoku game invented by Howard Garns is a special case of Latin Square which is usually 9 by 9 grids split into 9 smaller 3 b 3 boxes. The rule of the game is to fill every cell with one of the numbers from 1 to 9, so that each number appears exactly once in each row, column and 3 by 3 box- Wikipedia (2014).

In this paper, we adopt the idea of Sudoku game to develop another variation of Latin square design that adds one more restriction on randomization by blocking in 3 – dimensions (rows, columns and regions). It is worthy of note that our version of Latin square design only exists for a composite order P, P > 3. Of interest in this paper are the following fundamental questions:

- ✓ Can one construct a 3RR - Latin square design of a composite order P such that each of the P treatments appear only once in a row, only once in a column and only once in a region?
- ✓ What is the suitable statistical model for the 3RR – Latin Square design?
- ✓ How can the parameters in the model be estimated?
- ✓ How can the ANOVA be performed?

The answers to the above questions are provided in the following sections.

## II. CONSTRUCTION OF A 3RR - LATIN SQUARE DESIGN

We define a 3RR - Latin square design as an experimental design in which P treatments are arranged in a P x P array such that each treatment appears only once in a row, only once in a column and only once in a region. A region consists of P cells obtained by partitioning the P x P array in such a way that all the treatments are duly represented in the P regions. In the following examples, the regions are demarcated with different colors.

### A. 3RR - LATIN SQUARE OF ORDER 4

Suppose four treatments denoted by A, B, C and D are to be arranged in four rows, four columns and four regions such that each treatment appears once in a row, once in a column and once in a region. A typical 4 x 4 Latin square satisfying these conditions is shown below

| | | | |
|---|---|---|---|
| A | B | C | D |
| D | C | B | A |
| B | D | A | C |
| C | A | D | B |

*Table 1*

### B. 3RR - LATIN SQUARE OF ORDER 6

Here six treatments denoted by A, B, C, D, E and F are to be arranged in six rows, six columns and six regions such that each treatment appears once in a row, once in a column and once in a region. A typical example of the design is shown below.

| | | | | | |
|---|---|---|---|---|---|
| F | D | B | E | C | A |
| A | C | E | D | B | F |
| E | A | F | B | D | C |
| C | B | D | F | A | E |
| D | F | C | A | E | B |
| B | E | A | C | F | D |

*Table 2*

### C. 3RR - LATIN SQUARE OF ORDER 9

Here nine treatments denoted by A, B, C, D, E, F, G, H, and I are to be arranged in nine rows, nine columns and nine regions such that each treatment appears once in a row, once in a column and once in a region. A typical example of the design is shown below.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| G | I | B | A | D | E | C | F | H |
| H | D | C | B | F | G | E | I | A |
| E | F | A | C | H | I | G | B | D |
| B | C | I | H | E | A | D | G | F |
| F | G | E | D | C | B | A | H | I |
| D | A | H | G | I | F | B | E | C |
| I | E | G | F | A | D | H | C | B |
| C | B | D | I | G | H | F | A | E |
| A | H | F | E | B | C | C | D | G |

*Table 3*

## III. STATISTICAL MODEL

Based on the assumption of unit treatment additivity, a model for our 3RR – Latin square design [despite criticism by Srivastatva (1993, 1996), and Srivastava and Wang (1998)] is written as

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + \tau_l + e_{ijkl};\ i, j, k, l = 1, 2, \cdots, P \qquad (1)$$

Where,

$y_{ijkl}$ = Observation on experimental unit in row i, column j, region k to which treatment l is applied

$\alpha_i$ = Row i effect

$\beta_j$ = Column j effect

$\gamma_k$ = Region k effect

$\tau_l$ = Treatment l effect

$\mu$ = Overall mean

$e_{ijk}$ = Random error term which is assumed to be NID $(0, \sigma^2)$

## IV. ESTIMATION OF PARAMETERS

It is important to note that in a 3RR - Latin Square there are only $p^2$ experimental units to be used in the experiment instead of $p^4$ possible experimental units needed in a complete four way layout. Thus the use of 3RR - Latin Square design results in the savings in observations by a factor $1/p^2$ observation over the complete four layouts.

From equation (1), the sum of squares of errors is

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} e_{ijkl}^2 = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} (y_{ijkl} - \mu - \alpha_i - \beta_j - \gamma_k - \tau_l)^2 \qquad (2)$$

Differentiating equation (2) with respect to $\mu$, $\alpha_i$, $\beta_j$, $\Upsilon_k$ and $\tau_l$, respectively, and equating to zero, we obtain the following system of equations:

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p^2\mu - p\sum_{i=1}^{p}\alpha_i - p\sum_{j=1}^{p}\beta_j - p\sum_{k=1}^{p}\gamma_k - p\sum_{l=1}^{p}\tau_l = 0 \qquad (3)$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p\mu - p\alpha_i - p\sum_{j=1}^{p}\beta_j - p\sum_{k=1}^{p}\gamma_k - p\sum_{l=1}^{p}\tau_l = 0 \qquad (4)$$

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$$\sum_{i=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p\mu - p\sum_{i=1}^{p}\alpha_i - p\beta_j - p\sum_{k=1}^{p}\gamma_k - p\sum_{l=1}^{p}\tau_l = 0 \quad (5)$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p\mu - p\sum_{i=1}^{p}\alpha_i - p\sum_{j=1}^{p}\beta_j - p\gamma_k - p\sum_{l=1}^{p}\tau_l = 0 \quad (6)$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} y_{ijkl} - p\mu - p\sum_{i=1}^{p}\alpha_i - p\sum_{j=1}^{p}\beta_j - p\sum_{k=1}^{p}\gamma_k - p\tau_l = 0 \quad (7)$$

Assuming $\sum_{i=1}^{p}\alpha_i = \sum_{j=1}^{p}\beta_j = \sum_{k=1}^{p}\gamma_k = \sum_{i=1}^{p}\tau_l = 0$,

Equations (3) through (7) reduce to.

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p^2\mu = 0 \quad (8)$$

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p\mu - p\alpha_i = 0 \quad (9)$$

$$\sum_{i=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p\mu - p\beta_j = 0 \quad (10)$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} y_{ijkl} - p\mu - p\gamma_k = 0 \quad (11)$$

$$\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} y_{ijkl} - p\mu - p\tau_l = 0 \quad (12)$$

Solving equations (8) through (12) simultaneously, yields the following estimates of parameters

$$\hat{\mu} = \bar{y} \quad (13)$$

$$\hat{\alpha}_i = \bar{y}_{i...} - \bar{y},\ i = 1, 2, \ldots, p \quad (14)$$

$$\hat{\beta}_j = \bar{y}_{.j..} - \bar{y},\ j = 1, 2, \ldots, p \quad (15)$$

$$\gamma_k = \bar{y}_{..k.} - \bar{y},\ k = 1, 2, \ldots, p \quad (16)$$

$$\tau_l = \bar{y}_{...l} - \bar{y},\ l = 1, 2, \ldots, p \quad (17)$$

## V. ANALYSIS OF VARIANCE

The Analysis of Variance consist of partitioning the total sum of squares ($SST_0$) into its component parts, the sum of squares for row ($SSR_O$), sum of squares for column (SSC), sum of squares for region ($SSR_e$), sum of squares for treatment (SST) and sum of squares for error (SSE).

That is,

$$SST_0 = SSR_O + SSC + SSR_E + SST + SSE \quad (18)$$

Where

$$SST_O = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl}^2 - CF \quad (19)$$

with degrees of freedom, $v = p^2 - 1$

$$SSR_O = \frac{1}{p}\sum_{i=1}^{p} y_{i...}^2 - CF \quad (20)$$

with degrees of freedom, $v = p - 1$

$$SSC = \frac{1}{p}\sum_{j=1}^{p} y_{.j..}^2 - CF \quad (21)$$

with degrees of freedom, $v = p - 1$

$$SSR_E = \frac{1}{p}\sum_{k=1}^{p} y_{..k.}^2 - CF \quad (22)$$

with degrees of freedom, $v = p - 1$

$$SST = \frac{1}{p}\sum_{l=1}^{p} y_{...l}^2 - CF \quad (23)$$

with degrees of freedom, $v = p - 1$

$$SSE = SST_0 - SSR_0 - SSC - SSR_E - SST \quad (24)$$

with degrees of freedom, $v = (p - 1)(p - 3)$

$$CF = \left(\frac{\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{ijkl}}{p}\right)^2 \quad (25)$$

is a correction factor.

Following the general principle, it is easy to construct the ANOVA table for our design to enable us test the following hypotheses

$H_0 : \alpha_i = 0$, for all i

$H_0 : \beta_j = 0$, for all j

$H_0 : \gamma_k = 0$, for all k

$H_0 : \tau_l = 0$, for all l

The ANOVA table is shown below

| Source of variation | Degree of freedom | Sum of Squares | Mean Square | F-ratio |
|---|---|---|---|---|
| Row | P – 1 | $SSR_0$ | $MSR_0$ | $F_{RO}$ |
| Column | P – 1 | SSC | MSC | $F_C$ |
| Region | P – 1 | $SSR_E$ | $MSR_E$ | $F_{RE}$ |
| Treatment | P – 1 | SST | MST | $F_T$ |
| Error | (P – 1)(p – 3) | SSE | MSE | |
| Total | $P^2 – 1$ | $SST_0$ | | |

*Table 4*

In the above ANOVA table, the Mean squares and the F – ratios are obtained as follows

$$MSR_O = \frac{SSR_O}{P - 1}$$

$$MSC = \frac{SSC}{P - 1}$$

$$MSR_E = \frac{SSR_E}{P-1}$$

$$MST = \frac{SST}{P-1}$$

$$MSE = \frac{SSE}{(P-1)(P-3)}$$

$$F_{Ro} = \frac{MSR_0}{MSE}$$

$$F_C = \frac{MSC}{MSE}$$

$$F_{RE} = \frac{MSR_E}{MSE}$$

$$F_T = \frac{MST}{MSE}$$

$H_o$ is rejected at α% level of significance if the calculated F – ratios are greater than $F_\alpha$ with (p – 1) numerator degree of freedom and (p – 1)(p – 3) denominator degree of freedom.

## VI. SUMMARY AND CONCLUSION

One principle of experimental design is randomization which is often employed to reduce experimental errors. The existing Latin square reduced the experimental error by restricting randomization to rows and columns. For further reduction in error we restrict randomization to three dimensional, not as in Graeco Latin square, but by considering rows, columns and regions as in Sudoku game. Statistical model for the design has been developed, parameters in the model estimated and the Analysis of Variance performed. Construction of 3RR – Latin square are shown for order 4, 6 and 9.

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