Invariant Submanifold Of $\psi(6,4,2)$ Structure Manifold

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Abstract: In this paper, we have studied various properties of a $\psi(6,4,2)$ structure manifold and its invariant submanifold. Under two different assumptions, the nature of induced structure $\psi$, has also been discussed.

Keywords: Invariant submanifold, Nijenhuis tensor, projection operators and complementary distributions.

I. INTRODUCTION

Let $V^m$ be a $C^\infty$ m-dimensional Riemannian manifold imbedded in a $C^\infty$ n-dimensional Riemannian manifold $M^n$, where $m < n$. The imbedding being denoted by $f: V^m \longrightarrow M^n$.

Let $B$ be the mapping induced by $f$ i.e. $B = df$.

Then $B: T(V) \longrightarrow T(M)$ is an isomorphism. The set of all vectors normal to $f(V)$ forms a vector bundle over $f(V)$, which we shall denote by $N(V)$. We call $N(V)\times M$ the normal bundle of $V^m$.

The vector bundle induced by $f$ from $N(V)\times M$ is denoted by $N(V)\times f(V)$.

Thus $\zeta_0(V)$ is the space of all $C^\infty$ functions defined on $V^m$ while an element of $\eta^1_0(V)$ is a $C^\infty$ vector field normal to $V^m$ and an element of $\zeta^1_0(V)$ is a $C^\infty$ vector field tangential to $V^m$.

Let $\bar{X}$ and $\bar{Y}$ be vector fields defined along $f(V)$ and $\tilde{X}, \tilde{Y}$ be the local extensions of $\bar{X}$ and $\bar{Y}$ respectively. Then $[\tilde{X}, \tilde{Y}]$ is a vector field tangential to $M^n$ and its restriction $[\tilde{X}, \tilde{Y}]/f(V)$ to $f(V)$ is determined independently of the choice of these local extension $\tilde{X}$ and $\tilde{Y}$. Thus $[\tilde{X}, \tilde{Y}]$ is defined as

$$[\tilde{X}, \tilde{Y}] = [\tilde{X}, \tilde{Y}]/f(V)$$

Since $B$ is an isomorphism

$$[BX, BY] = B[X, Y]$$

for all $X, Y \in \zeta_0(V)$

Let $G$ be the Riemannain metric tensor of $M^n$, we define $g$ and $g^*$ on $V^m$ and $N(V)$ respectively as

$$g (X_1, X_2) = \tilde{G} (BX_1, BX_2) f$$

and

$$g^* (N_1, N_2) = \tilde{G} (CN_1, CN_2)$$

For all $X_1, X_2 \in \zeta_0^1(V)$ and $N_1, N_2 \in \eta_0^1(V)$

It can be verified that $g$ and $g^*$ are the induced metrics on $V^m$ and $N(V)$ respectively.
Let \( \tilde{\nabla} \) be the Riemannian connection determined by \( \tilde{\nabla} \) in \( M \), then \( \tilde{\nabla} \) induces a connection \( \nabla \) in \( f(V) \) defined by

\[
\nabla_{\tilde{X}} \tilde{Y} = \tilde{\nabla}_{\tilde{X}} \tilde{Y} / f(V)
\]

where \( \tilde{X} \) and \( \tilde{Y} \) are arbitrary \( C^\infty \) vector fields defined along \( f(V) \) and tangential to \( f(V) \).

Let us suppose that \( M^n \) is a \( C^\infty \) \( \tilde{\nabla} \) \( (6,4,2) \) structure manifold with structure tensor \( \tilde{\nabla} \) of type (1.1) satisfying

\[
\tilde{\nabla}^6 + \tilde{\nabla}^4 + \tilde{\nabla}^2 = 0
\]

Let \( \tilde{l} \) and \( \tilde{m} \) be the complementary distributions corresponding to the projection operators

\[
\tilde{l} = \tilde{\nabla}^6, \quad \tilde{m} = I - \tilde{\nabla}^6
\]

where I denotes the identity operator.

From (1.6) and (1.7), we have

\[
\begin{align*}
\tilde{l} + \tilde{m} &= I \\
\tilde{l}^2 &= \tilde{l} \\
\tilde{m}^2 &= \tilde{m} \\
\tilde{l} \tilde{m} &= \tilde{m} \tilde{l} = 0
\end{align*}
\]

Let \( D_l \) and \( D_m \) be the subspaces inherited by complementary projection operators \( l \) and \( m \) respectively.

We define

\[
\begin{align*}
D_l &= \{ X \in T_p(V): IX = X, mX = 0 \} \\
D_m &= \{ X \in T_p(V): mX = X, IX = 0 \}
\end{align*}
\]

Thus \( T_p(V) = D_l + D_m \)

Also \( Ker l = \{ X: IX = 0 \} = D_m \)

\( Ker m = \{ X: mX = 0 \} = D_l \)

at each point \( p \) of \( f(V) \).

II. INVARIANT SUBMANIFOLD OF \( \tilde{\nabla} \) \( (6,4,2) \) STRUCTURE MANIFOLD

We call \( V^m \) to be invariant submanifold of \( M^n \) if the tangent space \( \tilde{T}^p(f(V)) \) of \( f(V) \) is invariant by the linear mapping \( \tilde{\nabla} \) at each point \( p \) of \( f(V) \). Thus

\[
\begin{align*}
\tilde{\nabla}^6 BX &= \tilde{\nabla}X \quad \text{for all } X \in \mathfrak{z}^1_0(V), \quad \text{and } \tilde{\nabla} \text{ being a} \quad (1,1) \text{tensor field in } V^m.
\end{align*}
\]

**THEOREM (2.1):** Let \( \tilde{N} \) and \( N \) be the Nijenhuis tensors determined by \( \tilde{\nabla} \) and \( \tilde{\nabla} \) in \( M^n \) and \( V^m \) respectively, then

\[
\tilde{N}(BX, BY) = BN(X, Y), \quad \text{for all } X, Y \in \mathfrak{z}^1_0(V)
\]

**PROOF:** We have, by using (1.2) and (2.1)

\[
\tilde{N}(BX, BY) = [\tilde{\nabla}BX, \tilde{\nabla}BY] + \tilde{\nabla}^2[BX, BY] - \tilde{\nabla}[\tilde{\nabla}BX, BY] - \tilde{\nabla}B[\tilde{\nabla}BX, BY]
\]

III. DISTRIBUTION \( \tilde{\mathbf{M}} \) NEVER BEING TANGENTIAL TO \( f(V) \)

**THEOREM (3.1):** if the distribution \( \tilde{\mathbf{M}} \) is never tangential to \( f(V) \), then

\[
\tilde{m}(BX) = 0 \quad \text{for all } X \in \mathfrak{z}^1_0(V) \quad \text{and the induced structure } \tilde{\nabla} \text{ on } V^m \text{ satisfies}
\]

\[
\tilde{\nabla}^4 + \tilde{\nabla}^2 + I = 0 \quad \text{Thus } \tilde{\nabla} \text{ is } (4,2,0)
\]

**PROOF:** if possible \( \tilde{m}(BX) \neq 0 \). From (2.1) we get

\[
\tilde{\nabla}^2 BX = \tilde{\nabla}^2 X \quad \text{etc from (1.7) and (3.3)}
\]

\[
\tilde{m}(BX) = (I - \tilde{\nabla}^6) BX
\]

\[
= (I + \tilde{\nabla}^6 + \tilde{\nabla}^4) BX
\]

\[
= BX + \tilde{\nabla}^2 X + \tilde{\nabla}^2 X
\]

This relation shows that \( \tilde{m}(BX) \) is tangential to \( f(V) \), which contradicts the hypothesis. Thus \( \tilde{m}(BX) = 0 \). Using this result in (3.4) and remembering that \( B \) is an isomorphism, We get

\[
\tilde{\nabla}^4 + \tilde{\nabla}^2 + I = 0
\]

**THEOREM (3.2):** Let \( \tilde{\mathbf{M}} \) be never tangential to \( f(V) \).

\[
\tilde{N}(BX, BY) = 0
\]

**PROOF:** We have

\[
\tilde{N}(BX, BY) = [\tilde{\nabla}BX, \tilde{\nabla}BY] + \tilde{\nabla}^2[BX, BY] - \tilde{\nabla}[\tilde{\nabla}BX, BY] - \tilde{\nabla}B[\tilde{\nabla}BX, BY]
\]

**THEOREM (3.3):** Let \( \tilde{\mathbf{M}} \) be never tangential to \( f(V) \).

\[
\tilde{N}(BX, BY) = 0
\]

**PROOF:** We have

\[
\tilde{N}(BX, BY) = [\tilde{\nabla}BX, \tilde{\nabla}BY] + \tilde{\nabla}^2[BX, BY] - \tilde{\nabla}[\tilde{\nabla}BX, BY] - \tilde{\nabla}B[\tilde{\nabla}BX, BY]
\]

**THEOREM (3.4):** Let \( \tilde{\mathbf{M}} \) be never tangential to \( f(V) \).

Define

\[
\begin{align*}
\tilde{\mathbf{H}}(\tilde{X}, \tilde{Y}) &= \tilde{N}(\tilde{X}, \tilde{Y}) - \tilde{N}(\tilde{m}X, \tilde{Y}) - \tilde{N}(\tilde{X}, \tilde{m}Y) \\
&+ \tilde{N}(\tilde{m}X, \tilde{m}Y)
\end{align*}
\]

For all \( \tilde{X}, \tilde{Y} \in \mathfrak{z}^1_0(M) \), then
\[(3.11) \quad \mathbf{H} (BX, BY) = BN (X,Y)\]

**PROOF:** Using \( \mathbf{X} = BX, \mathbf{Y} = BY \) and (2.2), (3.1) in (3.10) We get (3.11).

**IV. DISTRIBUTION** \( \mathbf{M} \) **ALWAYS BEING TANGENTIAL TO** \( f(V) \)

**THEOREM (4.1):** Let \( \mathbf{M} \) be always tangential to \( f(V) \), then
\[
\begin{align*}
(4.1) & \quad \mathbf{m} (BX) = Bm X \\
(4.2) & \quad \mathbf{l} = \psi^6 X
\end{align*}
\]

**PROOF:** from (3.4), We get (4.1) (a). Also
\[
\begin{align*}
(4.3) & \quad \mathbf{B}lX = B \psi^6 X \\
(4.4) & \quad \mathbf{B}lX = \psi^6 BX = \mathbf{l} (BX),
\end{align*}
\]

**THEOREM (4.2):** Let \( \mathbf{M} \) be always tangential to \( f(V) \), then \( l \) and \( m \) satisfy
\[
\begin{align*}
(4.5) & \quad (a) \quad l + m = l \\
(4.6) & \quad (b) \quad lm = ml = 0 \\
(4.7) & \quad (c) \quad l^2 = l \\
(4.8) & \quad (d) \quad m^2 = m \\
\end{align*}
\]

**PROOF:** Using (1.8) and (4.1) We get the results.

**THEOREM (4.3):** If \( \mathbf{M} \) is always tangential to \( f(V) \), then
\[
\begin{align*}
(4.9) & \quad \psi^6 + \psi^4 + \psi^2 = 0
\end{align*}
\]

**PROOF:** From (2.1)

**REFERENCES**


[6] K. Yano: On a structure defined by a tensor field \( f \) of the type (1,1) satisfying \( f^3 + g = 0 \). Tensor N.S., 14 (1963), 99-109.