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Exponential Utility Optimization Of An Investor's Optimal Portfolios, Under Constant Elasticity Of Variance Model

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***Abstract:** Optimal portfolio of an investor is studied, in this work, when there are taxes, dividends, transaction costs and a risk-free asset with time varying rate of returns under constant elasticity of variance model. The Hamilton-Jacob–Bellman (HJB) equation associated with the optimization problem is obtained using the Ito's lemma. Explicit solution of the exponential utility maximization is obtained. It is found that the optimal investment of the investor is dependent on horizon. Also found is that the investment is the risky asset and is increased by a fraction when transaction costs and taxes are charged on the total investment of the investor. The investor should take the horizon and wealth dependency of his investment into consideration when making investment decisions.*

***Keywords:** Constant elasticity of variance, exponential utility function, optimal portfolio, transaction costs, taxes, dividends.*

***MSC 2010:** 91G10, 91G80*

I. INTRODUCTION

Optimal investment problem of utility maximization is an important issue in mathematical finance and has drawn attention of many researchers after Merton's work in which he proposed the stochastic control approach to study the investment problem for the first time Merton [1]. Other contributors followed in several dimensions; Pliska [2], Karatzas [3] adapted the martingale approach to investment problems of utility maximization. Zhang [4] investigated the utility maximization problem in an incomplete market using the martingale approach.

Recently, more and more researches study the utility maximization problem using stochastic control theory, (Devolder et al.[5], Yang and Zhang [6] and Wang [7]).

Most of the works mentioned above have the price processes of risky assets assumed to follow the geometric Brownian motion (GBM), which implies the volatility of risky asset is constant and deterministic which empirical evidence contradicted. Therefore, it became imperative that a model with stochastic volatility is more practical.

The constant elasticity of variance (CEV) model is a stochastic volatility model and is a natural extension of the GBM model. The CEV model was proposed by Cox and Ross [8] and has the ability of capturing the implied volatility skew (Li et al.[9]). Furthermore, this model is analytically tractable in comparison with other SV models. At first, the CEV model was usually applied to calculate the theoretical price, sensitivities and implied volatility of options, Davydov and Linetsky [10], Jones [11]. Recently, Xiao [12] and Gao [13], [14] have begun to apply the CEV model to investigate the utility maximization problem for a participant in the defined contribution pension plan. Gu [15] used the CEV model for studying the optimal investment and reinsurance problem of utility maximization. Lin and Li [16] considered an optimal reinsurance investment problem for an insurer with jump diffusion risk process under the CEV model. Zhao and Rong

[17] studied the portfolio selection problem with multiple risky assets under the CEV model.

Li and Ng [18] considered the mean-variance formulation in multi-period framework and first presented an embedding technique to obtain the analytical solution of the efficient strategy and efficient frontier. Zhou and Li [19] studied a continuous-time mean-variance portfolio selection model that is formulated as a bi-criteria optimization problem and the problem was seen as a class of auxiliary stochastic linear-quadratic (LQ) problems. Vigna and Haberman [20] analysed the financial risk in a DC pension scheme and found an optimal investment strategy. Gao [13] studied the portfolio optimization of DC pension fund under a CEV model and obtain the closed-form solution of the optimal investment strategy in power and exponential utility case. Li et al.[9] studied the optimal investment problem for utility maximization with taxes, dividends and transaction costs under the CEV model and obtained explicit solutions for the logarithmic, exponential and quadratic utility functions.

Jung [21] gave an explicit optimal investment strategy which maximizes the expected HARA utility of the terminal wealth under the CEV model. In Gu and Guo [22], optimal strategies and optimal value functions are obtained under a CEV model on the condition that the insurer can purchase excess-of-loss reinsurance.

These researches of optimal investment problem under the CEV model generally suppose that there are no taxes, dividends and transaction costs, which are not practical. To make our model more realistic, we consider taxes, dividends and transaction costs for optimal investment problem under the CEV model.

The investor aims to maximize the expected utility of his/her terminal wealth and is allowed to invest in a risk-free asset and a risky asset. In addition, we study this problem for logarithmic utility, exponential utility and quadratic utility, respectively. By applying the method of stochastic optimal control, the Hamilton-

Jacobi-Bellman (HJB) equation associated with the optimization problem is established and transformed into a complicated non-linear partial differential equation (PDE). Due to the difficulty of solution characterization, we use power transform and variable change technique proposed by Cox [23] to simplify the PDE and obtain the explicit solutions for the logarithmic and exponential utility functions. For the quadratic utility function, we use the Legendre transform and dual theory to solve the HJB equation.

Some studies about the portfolio selection problems with stochastic interest rate occurred. Korn and Kraft [24] used a stochastic control approach to deal with portfolio selection problems with stochastic interest rate and proved a verification theorem.

Deelstra et al.[25] studied the optimal investment problems for DC pension fund in a continuous-time framework and assumed that the interest rates follow the affine dynamics, including the CIR model and the Vasicek model. Chang et al.[26] studied an asset and liability management problem with stochastic interest rate in which the interest rate was assumed to be an affine interest rate model. Chang et al. [27] investigated an investment and consumption problem with stochastic interest rate, in which interest rate was assumed to follow the Ho-Lee model and be correlated with stock price and derived optimal strategies for power and logarithm utility function. Gao [13] investigated the portfolio problem of a pension fund management in a complete financial market with stochastic interest rate. Chang and Lu [28] studied an asset and liability management problem with CIR interest rate dynamics and obtained the closed form solutions to the optimal investment strategies by applying dynamic programming principle and variable change technique. Boulier et al. [29] obtained the optimal strategy for DC pension management with stochastic interest rate.

This paper intends to find the optimal investment strategy of the optimal portfolio for the financial market, in which interest rate of the risk-free asset is a linear function of time and the risky asset assumed to follow constant elasticity of variance (CVE) model and look into the variation that may occur when transaction costs and taxes are charged on the risky asset only and total investment.

The rest of the paper is organized as followed. In section 2, the introduction of the financial market is done and the wealth process established. Section 3 presents the optimization criterion and the derivation of the HJB equation. Also the introduction of the utility function and explicit solution of the optimal portfolio optimization problem obtained. Section 4 concludes the paper.

II. MODEL FORMULATION AND THE MODEL

Assuming an investor trades two assets in a financial market: a risky asset (stock) and a risk free asset (bond) that has a rate of returns that is a linear function of time. The dynamics of the price of the risk free asset denoted by $B(t)$ is given by

$$dB(t) = [\alpha + \beta t]B(t)dt\,; \;\; B(0) = 1, \qquad (1)$$

(Osu and Ihediohia, [30]), and that of the risky asset described by the Constant elasticity variance (CEV) model (Damping et al. Xiao et al. [31]; Gao, [14]).

$$\mathrm{dS}(\mathrm{t}) = \mathrm{S}(\mathrm{t})[\mu dt + bS^{\gamma}(t)dZ(t)], \qquad (2)$$

where $\mathrm{S}(\mathrm{t})$ denotes the price of the risky asset, α, β, μ, b and k are constants. μ is the appreciation rate of the risky asset as $\{Z(t){:}\, t > 0\}$ is a standard Brownian motion in a complete probability space $(\Omega, F, (F)_{t>0}, P)$. $F_{t>0}$ is the augmented filtration generated by the Brownian motion $Z(t)$. $bS^{\gamma}(t)$ is the instantaneous volatility and the elasticity, γ, a parameter that satisfies the general condition $\gamma \leq 0$, (Damping et al; 2013). If the elasticity parameter $\gamma = 0$, then equation (2), the constant elasticity of variance (CEV), reduces to a geometric Brownian motion .

Let $k(t)$ be the amount of money the investor puts in the risky asset at time t, then $[V(t) - k(t)]$ is the money amount he invested in the risk free asset, where $V(t)$ is his total money investment in both assets.

Suppose that in the financial market, taxes are charged at the rate λ, dividends at the rate $\emptyset$ and transaction costs of the rate θ.

ASSUMPTION

Dividends are paid only on the risky investment.
This study considers two cases;

- ✓ when transaction cost, and taxes are charged on the risky investment only.
- ✓ when transaction costs and taxes are charged on the total investment of the investor.

Corresponding to the trading strategy $k(t)$, the dynamics of the wealth process of the investor is given by the stochastic differential equation (SDE)

$$dV(t) = \frac{k(t)dS(t)}{S(t)} + \frac{[V(t)-k(t)]dB(t)}{B(t)} + [\emptyset - (\lambda+\theta)]\, k(t)dt, \tag{3}$$

when taxes and transaction costs are charged only on the risky investment, and

$$dV(t)\ \frac{k(t)dS(t)}{S(t)} + \frac{[V(t)-k(t)]dB(t)}{B(t)} + \emptyset\, k(t)dt - (\lambda+\theta)V(t)dt \tag{4}$$

when taxes and transaction costs are charged on the investor's total investment.

Substituting (1) and (2) into (3) and respectively obtains,

$$dV(t) = k(t)[\mu dt + bS^{\gamma}(t)dZ(t)] + [V(t)-k(t)][\alpha+\beta t]dt + [\phi - (\lambda+\theta)]k(t)dt, \tag{5}$$

which simplifies to

$$dV(t) = \{[(\mu+\phi) - (\alpha+\lambda+\beta t]k(t) + [\alpha+\beta t]V(t)\}dt + bS^{\gamma}(t)k(t)dZ(t). \tag{6}$$

Also, Substituting (1) and (2) into (4) and respectively obtains,

$$dV(t) = k(t)[\mu dt + bS^{\gamma}(t)dZ(t)] + [V(t)-k(t)][\alpha+\beta t]dt + \phi k(t)dt - (\lambda+\theta)V(t)dt \tag{7}$$

which simplifies to

$$dV(t) = \{[(\mu+\phi) - (\alpha+\beta t)]k(t) + [(\alpha+\beta t) - (\lambda+\theta)]V(t)\}dt + bS^{\gamma}(t)dZ(t)). \tag{8}$$

The quadratic variation equation of (6) and (8) is,

$$< dV(t) >= b^2 S^{2\gamma}(t)k^2(t)dt. \tag{9a}$$

Where,

$$\left.\begin{aligned} dt.dt = dt.dZ(t) = 0 \\ dZ(t).dZ(t) = 1 \end{aligned}\right\} \tag{9b}$$

Suppose the investor has a utility function $U(.)$, then the investor's problem can be written as

$$G(V,t;T) = Max_{k(t)} E[U(V)]|V^k(0) = v \tag{10}$$

subject to:

$$dV(t) = \{[(\mu+\emptyset) - (\alpha+\lambda+\theta+\beta t)]k(t) + [\alpha+\beta t]V(t)\}dt + bS^{\gamma}(t)dZ(t)$$

when the taxes and transaction costs are charged only on risky investment of the investor and

$$G(V,t;T) = Max_{k(t)} E[U(V)]|V^k(0) = v$$

subject to:

$$dV(t) = \{[(\mu+\emptyset) - (\alpha+\beta t)]k(t) + [(\alpha+\beta t) - (\lambda+\theta)]V(t)\}dt + bS^{\gamma}(t)dz(t)$$

when the taxes and transaction costs are charged on total investment of the investor.

III. THE OPTIMIZATION

The program for the investor's optimization problem is presented in this section. The study assumes that the investor has logarithmic utility preference.

That is,

$$U(V) = a - \frac{b}{f}e^{-fV}, \tag{11}$$

with absolute risk aversion,

$$\frac{-U''(V)}{U'(V)} = f, \tag{12}$$

where V is the wealth level of the investor. The aim of this study is to give an explicit solution to the investor's problem where his utility preference is the exponential function.

CASE 1: When the taxes and transaction costs are charged only on risky investment, the theorem below follows:

THEOREM 1:

The policy that maximizes the expected logarithmic utility of the investor when transaction costs and taxes are charged on risky investment only is to invest in the risky asset at each time, $t \le T$;

$$k^*(t) = \frac{[(\mu+\emptyset) - (\alpha+\theta+\lambda+\beta t)]}{fb^2 s^{2\gamma}(t)}$$

and the optimal value function;

$$G^*(V,t;T) = \left[a - \frac{b}{f}e^{-fV}\right]e^{-\int_t^T \eta(\tau)d\tau}.$$

PROOF:

The derivation of the Hamilton- Jacobi-Bellman (HJB) partial differential equation starts with the bellman equation

$$G(V,t:T) = Max_{k(t)} E[G(V',t;T)] \tag{13}$$

where V' denotes the amount of the investor's wealth at time, $t + \Delta t$. Hence,

$$Max_{k(t)} E^{(t,k)}[G(V',t;T) - G(V,t;T)] = 0 \tag{14}$$

The division of both sides of (14) by Δt and taking limit to zero gives the Bellman equation

$$Max_{k(t)} \frac{1}{dt} E[dG] = 0. \tag{15}$$

By Ito's lemma (Nie, [32]) that states;

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial V}dV + \frac{\partial^2 G}{2\partial V^2}(dV)^2 \tag{16}$$

Re-writing (16) by substituting (6) and (9) into gives,

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial V}\{[(\mu+\phi) - (\alpha+\beta t)]k(t) + [(\alpha+\beta t) - (\lambda+\theta)]V(t)\}dt + bS^{\gamma}(t)dZ(t)) + \frac{\partial^2 G}{2\partial V^2}b^2 S^{2\gamma}(t)k^2(t)dt \tag{17}$$

Applying (17) to (15) obtains,

$$Max_{k(t)}\frac{1}{dt}E\left\{\frac{\partial G}{\partial t}dt+\frac{\partial G}{\partial v}\{[(\mu+\phi)-(\alpha+\beta t)]k(t)+[(\alpha+\beta t)-(\lambda+\theta)]V(t)\}dt+bS^{\gamma}(t)dZ(t))+\frac{\partial^2 G}{2\partial V^2}b^2S^{2\gamma}(t)k^2(t)dt\right\}=0. \quad (18)$$

Equation (18) simplifies to the Hamilton-Jacob-Bellman, partial differential equation (P.D.E)

$$G_t+\{[(\mu+\emptyset)-(\alpha+\lambda+\theta+\beta t)]k(t)+[\alpha+\beta t]V(t)\}G_v+\frac{G_{VV}}{2}b^2S^{2\gamma}(t)k^2(t) \quad (19a)$$

where

$$E[dZ(t)]=0 \quad (19b)$$

Differentiating (19a) with respect to $k(t)$ gives

$$G_v[(\mu+\emptyset)-(\alpha+\lambda+\theta+\beta t)]+[2b^2S^{2\gamma}(t)k(t)]\frac{G_{vv}}{2}=0, \quad (20)$$

which simplifies to the optimal investment in the risky asset as:

$$k^*(t)=\frac{[(\alpha+\theta+\lambda+\beta t)-(\mu+\emptyset)]}{b^2s^{2\gamma}(t)}\frac{G_v}{G_{vv}} \quad (21)$$

Let

$$G(V,t;T)=h(t;T)\left[a-\frac{b}{f}e^{-fV}\right] \quad (22a)$$

be a solution to (19a) above with boundary condition

$$h(T;T)=1, \quad (22b)$$

then

$$G_t=h'(t,T)\left[a-\frac{b}{f}e^{-fV}\right];\ G_V=h(t;T)[be^{-fV}];G_{VV}=h(t;T)[-fbe^{-fV}]. \quad (23)$$

Using (23) in (21), obtains:

$$k^*(t)=\left[\frac{(\mu+\emptyset)-(\alpha+\theta+\lambda+\beta t)}{fb^2s^{2\gamma}(t)}\right]. \quad (24)$$

The optimal investment in the risky asset is dependent on horizon only, and the investor holds the risky asset as long as,

$$(\mu+\emptyset)-(\alpha+\theta+\lambda+\beta t)>0.$$

Now, applying (23) to (19) gives;

$$h'(t,T)\left[a-\frac{b}{f}e^{-fV}\right]+h(t,T)\left\{\begin{matrix}[be^{-fV}][(\mu+\emptyset)-(\alpha+\theta+\lambda+\beta t)]h(t;T)+\\ [\alpha+\beta t]V(t)-\frac{[fbe^{-fV}]}{2}[b^2S^{2\gamma}k^2(t)]\end{matrix}\right\}=0. \quad (25)$$

Further simplification of (25) gives:

$$h'(t;T)+\eta(t)h(t;T)=0, \quad (26a)$$

where

$$\eta(t)=\frac{1}{m}\left\{\begin{matrix}[be^{-fV}][(\mu+\emptyset)-(\alpha+\theta+\lambda+\beta t)]h(t;T)+\\ [\alpha+\beta t]V(t)-\frac{[fbe^{-fV}]}{2}[b^2S^{2\gamma}k^2(t)]\end{matrix}\right\}. \quad (26b)$$

and $$m=\left[a-\frac{b}{f}e^{-fV}\right] \quad (26c)$$

The solution of (26a) is;

$$\int_t^T\frac{h'(\tau,T)}{h(\tau,T)}d\tau=-\int_t^T\eta(\tau)d\tau, \quad (27)$$

from which obtains ;

$$h(t,T)=e^{-\int_t^T\eta(\tau)d\tau}, \quad (28)$$

and at the terminal time T

$$h(T,T)=1.$$

Therefore the optimal value function of the investor is

$$G(V,t;T)=\left[a-\frac{b}{f}e^{-fV}\right]e^{-\int_t^T\eta(\tau)d\tau}. \quad (29)$$

CASE 2:

The theorem below follows;

THEOREM 2:

The policy that maximizes the expected logarithmic utility of the investor when transaction costs and taxes are charged on the investor's total investment is to invest in the risky asset at each time t at;

$$k^*(t)=\frac{[(\mu+\emptyset)-(\alpha+\beta t)]}{fb^2s^{2\gamma}(t)}V(t)$$

and the optimal value function;

$$G^*(V,t;T)=\left[a-\frac{b}{f}e^{-fV}\right]e^{-\int_t^T\epsilon(\tau)d\tau}.$$

PROOF:

Applying (8) and (9) and going through the procedures of (13) to (18b), obtains the Hamillon- Jacobi- Bellman (H.J.B) equation;

$$G_t+G_v\{\{[\mu+\emptyset)-(\alpha+\lambda\beta t]k(t)+[(\alpha+\beta t)-(\lambda+\theta)]V(t)\}\}+\frac{G_{VV}}{2}\{b^2s^{2\gamma}(t)k^2(t)\}=0. \quad (30)$$

Differentiating (30) with respect to $k(t)$ obtains,

$$[\mu+\emptyset)-(\alpha+\lambda\beta t]G_v+[b^2s^{2\gamma}(t)k(t)]G_{VV}=0. \quad (31)$$

Equation (31) simplifies to the optimal investment in the risky asset to be,

$$k^*(t)=\frac{[(\alpha+\beta t)-(\mu+\emptyset)]G_v}{fb^2s^{2\gamma}(t)G_{VV}} \quad (32)$$

Applying (23) to (32) above gives,.

$$k^*(t)=\frac{[(\mu+\emptyset)-(\alpha+\beta t)]}{fb^2s^{2\gamma}(t)} \quad (33)$$

which is dependent on horizon only, and the investor holds the risky asset as long as ,

$$(\mu+\emptyset)-(\alpha+\beta t)>0. \quad (34)$$

Rewriting (30) using (23,) obtains,

$$lnV(t)h'(t,T)+\frac{h(t,T)}{V(t)}\{[(\mu+\emptyset)-(\alpha+\beta t)]k(t)+[(\alpha+\beta t)-(\lambda+\theta)]V(t)\}-\frac{h(t,T)}{2V^2(t)}[s^{2\gamma}k^2(t)]=0, \quad (35)$$

which reduces to,

$$h'(t,T)\left[a-\frac{b}{f}e^{-fV}\right]+h(t,T)\left\{\begin{matrix}[be^{-fV}]\{[(\mu+\emptyset)-(\alpha+\beta t)]+\\ [(\alpha+\beta t)-(\theta+\lambda)]V(t)\}-\frac{[fbe^{-fV}]}{2}[b^2S^{2\gamma}k^2(t)]\end{matrix}\right\}=0, \quad (36)$$

and further simplifies to,

$$h'(t,T)+\epsilon(t)h(t,T)=0 \quad (37a)$$

where

$$\epsilon(t)=\frac{1}{m}\left\{\begin{matrix}[be^{-fV}]\{[(\mu+\emptyset)-(\alpha+\beta t)]+\\ [(\alpha+\beta t)-(\theta+\lambda)]V(t)\}-\frac{[fbe^{-fV}]}{2}[b^2s^{2\gamma}k^2(t)]\end{matrix}\right\}, \quad (37b)$$

and $$m=\frac{1}{\left[a-\frac{b}{f}e^{-fV}\right]}. \quad (37c)$$

Therefore,

$$\int_t^T\frac{h'(\tau,T)}{h(\tau,T)}d\tau=-\int_t^T\epsilon(\tau)d\tau, \quad (38)$$

Giving,

$$h(t,T)=e^{-\int_t^T\epsilon(\tau)d\tau}, \quad (39)$$

which satisfies the boundary condition,

$$h(T,T)=1.$$

So the optimal value function for the investor's problem is given by,

$$G^*(V,t;T)=\left[a-\frac{b}{f}e^{-fV(t)}\right]e^{-\int_t^T\epsilon(\tau)d\tau} \quad (40)$$

COMPARISON:

The investments in the risky asset under the stated conditions are;

$$k_\pi{}^*(t) = \left[\frac{(\mu+\emptyset)-(\alpha+\theta+\lambda+\beta t)}{f b^2 s^{2\gamma}(t)}\right]$$

when transaction cost, and taxes are charged on the risky investment only, and,

$$k_v{}^*(t) = \frac{[(\mu+\emptyset)-(\alpha+\beta t)]}{f b^2 s^{2\gamma}(t)}$$

when transaction costs and taxes are charged on the total investment of the investor. Therefore

$$k_\pi{}^*(t) = \left[\frac{(\mu+\emptyset)-(\alpha+\theta+\lambda+\beta t)}{f b^2 s^{2\gamma}(t)}\right] = \left[\frac{(\mu+\emptyset)-(\alpha+\beta t))}{f b^2 s^{2\gamma}(t)}\right] - \frac{(\lambda+\theta)}{f b^2 s^{2\gamma}(t)} = k_v{}^*(t) - \frac{(\lambda+\theta)}{f b^2 s^{2\gamma}(t)} \quad (41)$$

Equation (41) shows that charging transaction costs and taxes on the investor's total investment warrants an increment in the in the investment in the risky asset.

Alternatively;

$$\frac{k_v{}^*(t)}{k_\pi{}^*(t)} = \frac{[(\mu+\emptyset)-(\alpha+\beta t)]}{[(\mu+\emptyset)-(\alpha+\theta+\lambda+\beta t)]} = \frac{[(\mu+\emptyset)-(\alpha+\beta t)]}{[(\mu+\emptyset)-(\alpha+\beta t)]-(\lambda+\theta)} \quad (42)$$

Let

$$(\lambda+\theta) = x[(\mu+\emptyset)-(\alpha+\beta t)], \quad (43)$$

where, $0 < x < 1$, then obtains,

$$\frac{k_v{}^*(t)}{k_\pi{}^*(t)} = \frac{[(\mu+\emptyset)-(\alpha+\beta t)]}{(1-x)[(\mu+\emptyset)-(\alpha+\beta t)]} = \frac{1}{(1-x)}. \quad (44a)$$

That is,

$$k_v{}^*(t) : k_\pi{}^*(t) = 1 : (1-x). \quad (44b)$$

This clearly shows that the amount invested in the risky asset when transaction costs and taxes are charged on the total investment needs to be increased for the investor's smooth operations.

IV. CONCLUSION

This study gives the optimization of the exponential utility of an investor when transaction costs, taxes, are charged on the risky investment only and the total investment, adopting the constant elasticity of variance (CEV) model to describe the dynamic movements of the risky asset's price.

Applying stochastic optional control the corresponding Hamilton-Jacobi-Bellman (HJB) equation obtained, using Ito's lemma for which an explicit solution is obtained considering when transaction costs and taxes are charged on the risky investment only and the total investment of the investor.

It is found that charging transaction costs and taxes on the total investment warranted an increase in the risky investment asset as compared to when these charges are on the risky investment only.

It is found that the investments are both horizon and wealth dependent which the investor must put into consideration when making investments policy decisions.

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