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Approximation Of Function By Fourier Series

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***Abstract:** This paper presents a Fourier series approach for the analysis of increasing function described by approximation theory. An approximation theory can be found out between two comparable functions. It's given an idea for suitable solution for the problem.*

***Keywords:** Fourier series, approximation, increasing function.*

I. INTRODUCTION

The approximation theory is very important and useful branch in mathematics. It was wierestrass who first introduce the theory of approximation. He proves that every continuous function on a compact interval i.e. [a,b] is uniformly approximate by polynomial which converges in [a,b] to f . i.e. for given $\varepsilon > 0$, there is a polynomial p(x) such that

$|f(x) - p(x)| < \varepsilon$

When we prove wierestrass theorem with the help of Fejer's theorem it is prove that for a function σ_n (x) uniformly to f(x).

Where σ_n (x) is Trigonometric polynomial.

Wierestrass also prove that corresponding theorem on approximation by mean of trigonometric sum. In 1821 the French mathematician A. L. Cauchy formulated the concept of convergence of infinite series. According to Cauchy:

An infinite series $\sum a_n$ is said to be convergent to sum S as $n\rightarrow\infty$ i.e. $\forall\ \varepsilon > 0\ \exists$ N $\in I^+$ such that

$|S_n - S| < \varepsilon$ for $n \geq N$

An infinite series $\sum a_n$ is said to be absolutely convergent if

$\sum |S_n - S| < \infty$

II. SOME DEFINITIONS

A. FOURIER SERIES APPROXIMATION

The Real Fourier Series is

$S(t) – ao + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b_k \cos\left(\frac{2\pi kt}{T}\right)$

For a real Fourier series, we can re-write Parseval's Theorem

$$\frac{1}{T}\int_0^T s^2\,(t)\,dt - a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} a_k^2 + b_k^2$$

B. RIEMANN INTEGRATION

Suppose that the function f is bounded on the interval [a,b] where a,b ϵ R ad a < b and consider a dissection

Δ : a = x0 < x1 < x2 ……….< xn = b of [a, b]

Then

Definition: (Riemann Sum) : The lower Riemann sum of f(x) corresponding to the dissection Δ is defined as following sum:

$s\,(f,\ \Delta) = \sum_{j=1}^{n} x_j - x_{j-1}\ \inf_{x\ \in|x_{j-1},x_j|} f(x)$

and the upper Riemann sum of f(x) corresponding to the dissection Δ is defined as following sum:

$S\,(f,\ \Delta) = \sum_{j=1}^{n} x_j - x_{j-1}\ \mathrm{Sup}_{x\ \in|x_{j-1},x_j|} f(x)$

C. LEBESGUE INTEGRATION

Suppose we are considering integrating function like $\chi(x)$, the characteristic function of set S ={ $x \in$ Q} $\subset$ R (i.e. $\chi_s(s) = 1$ If $x \in$ S and $\chi_s(s) = 0$ If $x \notin$ S) Or suppose we are considering real-valued measurements x of a phenomenon and wondering what is probability foe x to be rational number .Form a probabilistic theory, we know that if the measurement are distributed normally with a mean of μ and a standard deviation σ , then the probability is given by :

$$P_r\,[\,x \in Q\,] = \int_S \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx$$

And the Riemann integral is useless to evaluate this integral.

First, we have following lemmas for the proof of the theorem:

III. SOME IMPOTANT RESULTS

LEMMA 1: If $K_n(t)$ be the Fejer Kernel of order 1, then

$$K_n(t) = \begin{bmatrix} o(n) \;; & for\ 0 < t < \frac{1}{n} \\ o\left(\frac{1}{nt^2}\right); & for\ \frac{1}{n} < t < \delta \\ o\left(\frac{1}{n}\right); & for\ \delta < t < \pi \end{bmatrix}$$

LEMMA 2: Φ (t) = $\int_t^{\delta} \frac{\Phi(u)}{u^2}\, du = o[t^{\alpha}\,\psi(t)]$ where ψ(t) is increasing function of t, then $\Phi'(t) = \frac{\Phi(t)}{t^2}$

$$\int_0^t t^2\,\Phi'(t)\,dt = \int_0^t \Phi(t)\,dt$$

$$\int_0^t |\Phi(u)|\,du = \int_0^t u^2\,\Phi'(u)du$$

$$= [u^2\Phi(u)]t0 + \int_0^t 2u\,\Phi(u)du$$

$$= o[t^2 t^{\alpha}\,\psi(t)] + o[2u^{\alpha+2}\,\psi(u)du]$$

$$= o[t^{\alpha+2}\,\psi(t)] + o\left[t^{\alpha}\,\psi(t)\int_0^t 2u\,du\right]$$

$$= o[t^{\alpha+2}\,\psi(t)] + o\,[t^{\alpha+2}\,\psi(t)]$$

$$= o[t^{\alpha+2}\,\psi(t)]$$

i.e. $\int_0^t |\Phi(u)|\,du = o[t^{\alpha+2}\,\psi(t)]$

IV. MAIN RESULT

A major result is the following:

Let α > 0 and 0 < δ < π and if x is a point such that

$$\Phi(t) = \int_t^{\delta} \frac{\Phi(u)}{u^2}\,du = o[t^{\alpha}\,\psi(t)]$$

Where ψ(t) is increasing function of t then

$\sigma_n(x) - f(x) = o\,[\,n\text{-}(\alpha+1)\,\psi\left(\frac{1}{n}\right)]$

Where $t^{\alpha}\,\psi(t) \to 0$ as $t \to 0$

PROOF: Since $\sigma n\,(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)K_n(u)\,du$

Where $K_n(u)$ be the Fejer Kernel of order 1

$$\sigma_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi}[f(x+u) + f(x-u) - 2f(x)]K_n(u)\,du$$

$$= \frac{1}{\pi}\int_0^{\pi} \Phi(u)\,K_n(u)\,du$$

Where $\Phi(u) = f(x+u) + f(x-u) - 2f(x)$

$$\sigma_n(x) = \frac{1}{\pi}\int_0^{\pi} \Phi(u)\,Kn(u)\,du$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right] Kn(u)\,du$$

$= I_1 + I_2 + I_3$ (say)

CONSIDER I_1

$$I_1 = \frac{1}{\pi}\int_0^{\frac{1}{n}} \frac{|\Phi(t)|}{t^2}\,Kn(t)\,dt$$

$$= o\left[n\int_0^{\frac{1}{n}} \frac{|\Phi(t)|}{t^2}\,dt\right] \quad \text{by } lemma1$$

$$= o\,[\,n\,t^{\alpha+2}\,\psi(t)] \quad \text{by } lemma2$$

$$= o\left[n\,\frac{1}{n^{\alpha}n^2}\,\psi\left(\frac{1}{n}\right)\right]$$

$$I_1 = o\,[n^{-(\alpha+1)}\,\psi\left(\frac{1}{n}\right)] \quad \ldots\ldots. (1.1)$$

CONSIDER I_2

$$I_2 = \frac{1}{\pi}\int_{\frac{1}{n}}^{\delta} \frac{|\Phi(t)|}{t^2}\,Kn(t)\,dt$$

$$= o\left[\frac{1}{nt^2}\int_{\frac{1}{n}}^{\delta} \frac{|\Phi(t)|}{t^2}\,dt\right] \quad \text{by } lemma1$$

$$= o\left[\frac{1}{n\frac{1}{n^2}}\,t^{\alpha+2}\,\psi(t)\right] \quad \text{by } lemma2$$

$$= o\left[n\,\frac{1}{n^{\alpha}n^2}\,\psi\left(\frac{1}{n}\right)\right]$$

$$I_2 = o\,[n^{-(\alpha+1)}\,\psi\left(\frac{1}{n}\right)] \quad \ldots\ldots. (1.2)$$

CONSIDER I_3

$$I_3 = \frac{1}{\pi}\int_{\delta}^{\pi} |\Phi(u)|\,Kn(u)\,du$$

$$= o\left[\frac{1}{n}\int_{\delta}^{\pi} \frac{|\Phi(t)|}{t^2}\,dt\right] \quad \text{by } lemma1$$

$\Phi(u)$ is a lebesgue integrable and 0< δ < π, therefore it follows that

$$I_3 = o\,[n^{-(\alpha+1)}\,\psi\left(\frac{1}{n}\right)] \quad \ldots\ldots. (1.3)$$

Combining the relation (1.1), (1.2) and (1.3) we get

$$\sigma_n(x) - f(x) = o\,[\,n\text{-}(\alpha+1)\,\psi\left(\frac{1}{n}\right)]$$

This complete proof of theorem.

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