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Fundamentals And Concept Of Conformal Mappings

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***Abstract:** Complex analysis proves a useful tool for solving a wide variety of problems in engineering science — the analysis of ac electrical circuits, the solution of linear differential equations with constant coefficients, and the representation of wave forms, and so on. Theory of complex numbers consists of functions of complex numbers. They are defined in a similar way to functions of real numbers that is studied in calculus; the only difference is that they involve complex numbers rather than real numbers. A complex-valued function f(z) of the complex variable z is a rule that assigns to each complex number z in a set D to one and only one complex number w. We write w = f(z) and call w the image of z under w = f(z). To geometrically illustrate a function of a complex variable it is convenient to consider two different planes with rectangular coordinates. These planes are called the z-plane and the ω-plane. Functions of a complex variable can be illustrated graphically by indicating correspondences between sets of points in these two planes, i .e .the z-plane and the ω-plane. This review paper will just revise the basics of conformal mappings.*

***Keywords:** Complex variable, the z-plane and the ω-plane, conformal mapping*

I. INTRODUCTION

Mathematics is everywhere in every phenomenon, technology, observation, experiment etc. All we need to do is to understand the logic hidden behind. Since mathematical calculations give way to the ultimate results of every experiment, it becomes quite pertinent to analyse those calculations before making conclusions. Conformal Mapping is a mathematical mapping that transforms circles to circles and straight lines to straight lines. In mathematics, a conformal map is a function that preserves angles locally. In the most common case, the function has a domain and an image in the plane. More formally, a map

$$f : U \rightarrow V \text{ with } U, V \subseteq C^n$$

is called conformal (or angle-preserving) at a point u_0 if it preserves oriented angles between curves through u_0 with respect to their orientation (i.e. not just the magnitude of the angle). Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.

II. DEFINITION OF CONFORMAL TRANSFORMATION

Suppose the two curves C and C_1 in the Z-plane intersect at the point P and the corresponding curves C' and C_1' in the W-plane intersect at P'.If the angle of intersection of the curves at P is same as the angle of intersection of the curves at P'. In magnitude and sense then the transformation is said to be conformal.

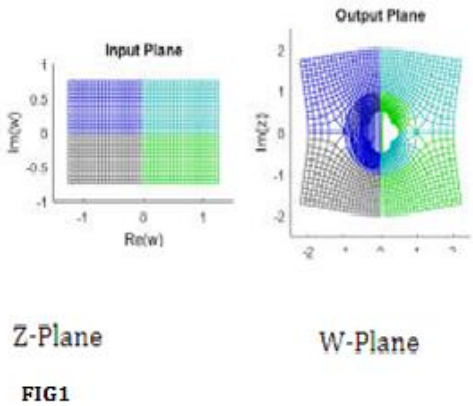

Figure 1

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THEOREM: The transformation effected by an analytical function

$w = f(z)$ is conformal at every point of the Z-Plane where

$f'(Z) \neq 0$

III. SPECIAL CONFORMAL TRANSFORMATIONS

✓ *TRANSFORMATION* $W = Z^2$:

Let $z = x + iy$ and $w = u + iv$

Consider the equation $w = z^2$,

So $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$

Comparing real and imaginary parts

$u = x^2 - y^2$ and= $2xy$ .... (1)

If u is constant (say a), then $x^2 - y^2 = a$ which is a rectangular hyperbola. Similarly if v is constant (say b) then $xy = \frac{b}{2}$ which also represents a rectangular hyperbola.

Hence a pair of lines u=a and v=b parallel to axes in the W-plane map into pair of orthogonal rectangular hyperbolae in the Z-plane.

Into what does the function map the rectangular area in the z plane bounded by the lines x = 1/2, x = 1, y = 1/2, y = 1. One can answer the question by asking what it maps the lines x = 1/2, x = 1, y = 1/2, y = 1 into. To answer that question we proceed as follows. We expand the function $w = z^2$.

We have the answer from FIG 2.

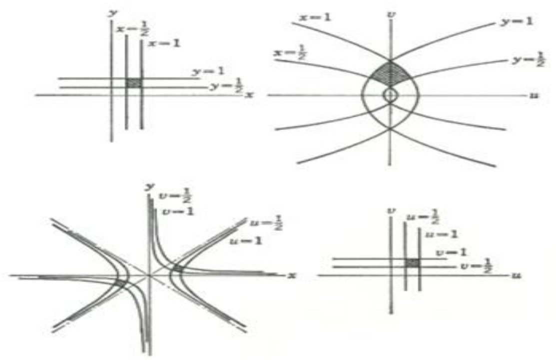

FIG2

Figure 2

✓ *TRANSFORMATION* $W = e^Z$:

Let $z = x + iy$, and $w = \rho e^{i\emptyset}$

Also by using equation of transformation $w = e^z$

We have $\rho e^{i\emptyset} = e^{x+iy} = e^x e^{iy}$

Equating both sides of the equation $\rho = e^x$..............eq.(2)

And $\emptyset = y$...............eq.(3)

Since we have already defined

e^{iy}= cos(y) + isin(y), we have

e^z= e^x (cos(y) + isin(y)).

Now if, and $w = \rho e^{i\emptyset}$

then w =$e^z = e^x e^{iy}$ implies that $\rho = e^x$ and $\emptyset = y$+ 2kπ,

where k = 0, ±1, ±2, ... In particular, if z lies on the vertical line x = c, that is, z = c + iy, then w traverses the circle of radius e c with centre at the origin as y passes through every interval of length 2π. That is, $w = e^z$ maps vertical lines onto circles centered at the origin, with the mapping repeating a countercockwise traversal of the circle an infinite number of times as y goes from −∞ to ∞. If z lies on the horizontal line y = c, that is z = x + ic, then w lies on the ray ∅ = c. In fact, w traverses this entire ray as x goes from −∞ to ∞. As a consequence of the above observations, $w = e^z$ maps a rectangle R = [a, b] × [c, d] in the z-plane onto a circular sector.

{ $w = \rho e^{i\emptyset}$:$e^a \le \rho \le e^b$,c ≤ ∅ ≤ d} in the w-plane.

For the infinite strip S = {z = x + iy: 0 ≤ y ≤ π}, $w = e^z$ maps S onto {w = u + iv: v ≥ 0, w ≠ 0}.

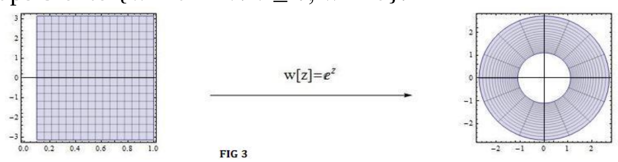

Figure 3

✓ *TRANSFORMATION* $W = \cosh z$:

Let $z = x + iy$ and $w = u + iv$

$u + iv = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$

So that

$u + iv = \cosh x \cos y$ and $v = \sinh x \sin y$

On eliminating x from these equations gives

$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1$....................eq(4)

And elimination of y gives

$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sin^2 y} = 1$......................eq(5)

Eq (4) shows that the lines parallel to x axis in the z-plane map into hyperbolae in the w-plane. Eq (5) shows that the lines parallel to y axis in the z-plane maps into ellipse in the w-plane. The rectangular region $a_1 \le x \le a_2, b_1 \le y \le b_2$ in the z-plane transforms into the w-plane bounded by corresponding hyperbolae and ellipses.

IV. APPLICATIONS

A large number of problems arising in fluid mechanics, Electrostatics, heat conduction, and many other physical situations can be mathematically formulated in terms of Laplace equation. i.e., all these physical problems reduce to solving the equation

$$\varphi_{xx} + \varphi_{yy} = 0$$

in a certain region D of the z plane. The function _(x, y), in addition to satisfying this equation also satis_escertain boundary conditions on the boundary C of the region D. From the theory of analytic functions we know that the real and the imaginary parts of an analytic function satisfy Laplace's equation. It follows that solving the above problem reduces to ending a function that is analytic in D and that satisfies certain boundary conditions on C. It turns out that the solution of this problem can be greatly simplified if the region D is either the upper half of the z plane or the unit disk.

V. CONCLUSION

There are different aspects of conformal mapping that can be used for practical applications though the essence remains

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the same: it preserves the angle and shapes but not the size. These properties of conformal mapping make it advantageous in complex situations. Various conformal techniques are applied and used in every branch of Engineering.

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