International Journal of Innovative Research and Advanced Studies (IJIRAS)
Volume 3 Issue 10, September 2016

ISSN: 2394-4404

A Batch Arrival Multi-Server Queueing Model With Finite Capacity Using Diffusion Approximation

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***Abstract:** In this paper we analyze a $G^X/G/m$ queueing model with finite capacity using diffusion approximation. Two boundaries associated with elementary return process have been imposed at 0 and N for the solution purpose. In this paper formulae for the number of customers, delay probability and mean queue length in the system have been derived for the steady-state distribution.*

***Keywords:** Multi-server, batch service, diffusion approximation.*

I. INTRODUCTION

Queueing system can be described as customers arriving for service if not immediately provided and if having waited for service, leaving the system after getting service.Queueing model is considered as bulk queues where arrival of customers is more than one, e.g. people going to theatre or restaurant, letters arriving at a post-office,Ships arriving at a port in convoy and so on. Also queueing models in which customers arrive in batches have been widely applied to many practical situations, such as computer networks, communication systems and production / manufacturing systems. Mathematical modeling of bulk arrival and batch service queues yields different performance measures to analyze production / manufacturing systems. We can fix the minimum and maximum capacity using optimization method. Al though bulk arrival multi- server queueing systems with general inter-arrival and service time distribution have been very useful to investigate the performance of various systems in the areas such as inventory systems, manufacturing, production, telecommunication and transportation, it is very difficult to find explicit results for these queueing systems. Many researchers used some approximation methods to overcome this difficulty, see, Hokstad (1978), kimura(1983 ,1994,2000), Miyazava (1986), Whitt(1993, 2005), Chen & shanthikumar (1994),Xiong & Altiok (2009), etc.

In queueing literature, multi-server queueing models based on diffusion approximation were considered by Stoyan (1976) , Kobayashi (1974), Takahashi (1977) , Hokstad (1978) , Halachmi and Franta (1978) , and Kimura (1983) , Chaudhry and Templeton (1983) provided several studied about a certain subclass of $GI^X/G/m$ queueing systems. Ohsone (1984) analysed $GI^X/G/m$ model by using diffusion approximation technique. Some more researchers who have been given greate contributions to multi-server queueing systems are Jain & sharma (1985,1986) , Wu (1990),Whitt(1982, 2004)

For finite capacity queueing models, approximation based on diffusion process have also been used by some authors, Hokstad (1978) , Nozaki and Ross (1978) considered M/G/c/N queueing model was analysed by Kimura et al. (1979), Gelenbe and Mitrani (1980). Sunaga et al. (1982) proposed G/G/c/N model by using the diffusion approximation technique. Yao and Buzacott (1985) developed a G/G/c/N queueing model for a flexible machining station. Modified diffusion approximation for GI/G/1 and GI/G/s system was proposed by Takahashi et al. (1987). Finite capacity queueing systems also got the attention of some more researchers e.g. Kimura (1995), (2003), Whitt(2004)

In this paper, we consider $G^X/G//N$ queueing model with finite capacity. Two boundaries associated with elementary return processes are imposed at origin and N for solution purpose.

II. FORMULATION AND DIFFUSION PROCESS

We consider the $G^X/G/m//N$ System with the following assumptions:

- ✓ Customers arrive in groups according to a general process with rate $\lambda(>0)$.
- ✓ The group sizes are positive integer valued i.i.d. random variables with a distribution (g_n, n= 1,2, . . .N).
- ✓ The distribution g_n has finite mean γ ($\geq$ 1) and variance σ_g^2 .
- ✓ Customers arriving in groups are served in order of their arrivals.
- ✓ The service times of customers identically and independentically distributed with mean service rate $1/\mu$.

The following notations are also used in this paper:-

Q(t) = the no of customers in the system at time t.

$P_n(t)$ = the probability that Q(t) is n.

σ_a^2 = the variance of inter-arrival time distribution

σ_s^2 = the variance of service time distribution

C_a = coefficient of variation of arrival times

C_s = coefficient of variation of service times.

Now we proceed to the diffusion approximation for our model.

Let { x(t): t ≥ 0 } be a homegeneous diffusion process in one dimension. Define the probability density function (p.d.f.) of X(t) by

$$p(x, t/x_o)\, dx = p\,\{ x \leq X(t) < x + dx \mid X(0)=x_o \} \qquad (1)$$

p (x, t/x_o) satisfies the forward Kolmogorov equation (See Cox and Miller, 1965).

$$\frac{1}{2}\frac{\partial^2}{\partial x}\{ a(x)\, p(x, t/x_o) \} - \frac{\partial}{\partial x}\{ b(x)\, p(x, \mid tx_o) \} = \frac{\partial p}{\partial t} \qquad (2)$$

where b(x) and a(x) are the infinitesimal mean and variance, respectively, defined by

$$b(x) = \lim_{\Delta t\to 0} E\,[X(T+\Delta t) - X(t) \mid X(t)\,] / \Delta t \qquad (3a)$$

$$a(x) = \lim_{\Delta t\to 0} Var\,[X(T+\Delta t) - X(t) \mid X(t)\,] / \Delta t \qquad (3b)$$

for steady state $p(x) = \lim_{t\to\infty} p(x, t/x_o)$ so that

equation (2) becomes

$$\frac{1}{2}\frac{d^2}{dx}\{a(x)\, p(x)\} - \frac{d}{dx}\{b(x)\, p(x)\} = 0 \qquad (4)$$

for our queueing system, we propose

$$b(x) = \lambda\gamma - ([x]^{\wedge}m)\,\mu \qquad (5a)$$

$$a(x) = \lambda\,(\sigma_a^2\,\gamma^2 + \sigma_g^2\,) + ([x]^{\wedge}\, m)\,\mu^3\,\sigma_s^2 \qquad (5b)$$

where [x] denotes the smallest integer not smaller than x, and ^ stands for minium. The positive integer [x]^ m just corresponds to the number of busy servers. For m$\geq$ 2, both b(x) and a(x) are piecewise continuous functions having (m-1) first order discontinuity points.

The diffusion process {X(t)} has two boundaries. Suppose n_0^{-1} and n_N^{-1} denote the mean holding times of { X(t)} at 0 and N, respectively. Since n_0^{-1} and n_N^{-1} can be derived exactly, therefore we make some assumptions. Suppose that the arrival of a customer makes system full and the system remains full until one of the server becomes idle. We approximate this time interval as the residual interdeparture time and therefore propose.

$$h_N = 2\, m\mu \,/\, (\, C_s^2 + 1)$$

Similarly, the system remains the departure of last customers in service, until another customers arrives.

This time interval is approximated as the residual inter-arrival time so that we propose

$$h_o = 2\,\lambda \,/\, (C_a^2 + 1)$$

Let p(x) be the probability density function (p.d.f.) of X(.) in the seady-state. Then it follows from Feller (1954) that p(x) satisfies the following equation

$$\frac{1}{2}\frac{d^2}{dx}\{ a(x)\, p(x)\} - \frac{d}{dx}\{ b(x)\, p(x) \} = -h_o\,\pi_o \sum_{K=1}^{N} g_K\, \delta(x-k) - h_N\, N\, \delta(x-N-1) \qquad (6)$$

where π_o denotes the probability mass at the origin and $\delta(.)$ denotes Dirac's delta function.

The boundary conditions at x=0 and at x=N are

$$\frac{1}{2}\frac{d}{dx}\{a(x)\, p(x)\} - \{b(x)\, p(x)\}\,\Big|_{x=0} = h_o\,\pi_o \qquad (7a)$$

and $\frac{1}{2}\frac{d}{dx}\{ a(x)\, p(x)\} - \{b(x)\, p(x)\}\,\Big|_{x=N} = -\pi_o\, h_{oN}$ (7b)

Also $\lim_{x\to 0} p(x) = 0$ (8a)

$$\lim_{x\to N} p(x) = 0 \qquad (8b)$$

and $\pi_o + \int_0^N p(x)\, dx + \pi_N = 1$ (9)

III. THE SOLUTION OF THE DIFFUSION EQUATION

Taking integral on both sides of (6), we get

$$\frac{1}{2}\frac{d}{dx}\{ a(x)\, p(x)\} - b(x)\, p(x)$$

$$h_o\pi_o\left[1 - \sum_{K=1}^{N} g_K U(x-k)\right] - h_N N\, U\,(\,x - \overline{N-1}) \qquad (10)$$

where U(.) denotes the unit step function. We see that (7b) is also satisfied, if we let x→ N in (10)

Now we assume that for k-1$\leq$x<k (k= 1,2 . . . . N, x$\neq$ 0)

$b(x)=b_k$

$a(x)=a_k$

and $\overline{g_k} = p\,\{x \geq k\} = 1 - \sum_{i=1}^{k-1} g_i$

using equation (11), equation (10) can be rewrite as follow

$$\frac{1}{2}a_k\frac{dp_k}{dx} - b_k p_k = \begin{cases} \pi_o h_o & (0<x<1);\ k=1 \\ \pi_o h_o \overline{g_k} & (k-1 \leq x < k);\ k = 2,3\ldots, N-1 \\ -\pi_N N & (N-1 \leq x < N);\ k=N \end{cases}$$

where $p_k = p_x(x)$ is the restriction of p(x) to the interval (k-1,k) of $R_+ (k = 1, \ldots N)$

for each k, a general solution of (12) can be derived, for k=1, we use boundary condition (8a). Because of continuity of p(x), we impose the smoothing condition.

$\lim_{x\to k-1} p_k(x) = p_{k-1}\,(k-1)$, for k=2,3,. . . . N (13)

Now the solution of (12) is given by

✓ for 0<x<1

$$p(x) = \begin{cases} \dfrac{\pi_o h_o}{b_1}\left(e^{\frac{x}{r_1}} - 1\right) & \text{if } b_k \neq 0 \\ \dfrac{2\,\pi_o h_o}{a_1}\, x & \text{if } b_k = 0 \end{cases}$$

✓ for $k-1 \leq x < k$, (k= 2,3 . . . .N-1)

$$p(x) = \begin{cases} \pi_o \left[\xi_k + \frac{h_o\overline{g_k}}{b_k} e^{\frac{x-k}{r_k}} - \frac{h_o\overline{g_k}}{b_k}\right] & \text{if } b_k \neq 0 \\ \pi_o \left[\xi_k + \frac{2h_o\overline{g_k}}{a_k}(x-k)\right] & \text{if } b_k = 0 \end{cases}$$

✓ for $N-1 \leq x < N$

$$p(x) = \begin{cases} \frac{\pi_N h_N}{b_N}\left(1 - e^{\frac{x-k}{r_k}}\right) & \text{if } b_k \neq 0 \\ \frac{2\pi_N h_N}{a_N}(N - x) & \text{if } b_k = 0 \end{cases}$$

where $k = \frac{a_k}{2b_k}$ (K=1,2. . . . . .,N)

$\xi_0 = 0$

$$\xi_k = \begin{cases} \left(\xi_{k-1} + \frac{h_o\overline{g_k}}{b_k}\right) e^{\frac{1}{r_k}} - \frac{h_o\overline{g_k}}{b_k} & \text{if } b_k \neq 0 \\ \xi_{k-1} + \frac{h_o\overline{g_k}}{b_k} & \text{if } b_k = 0 \end{cases}$$

(k= 1,2 . . .N)

$$\pi_{N} = \begin{cases} \frac{\pi_o b_N}{h_N(1-e^{r_N})}\xi_{N-1} & \text{if } b_k \neq 0 \\ \frac{\pi_o a_N}{2h_N}\xi_{N-1} & \text{if } b_k = 0 \end{cases}$$

We can then approximate the steady state probabilities as follows:

$$\pi_k \int_{k-1}^{k} p(x)dx = \begin{cases} \pi_o\left[r_k(\xi_k - \xi_{k-1}) - \frac{h_o\overline{g_k}}{b_k}\right] & \text{if } b_k \neq 0 \\ \pi_o\left(\xi_k - \frac{h_o\overline{g_k}}{b_k}\right) & \text{if } b_k = 0 \end{cases}$$

Where (k= 1,2 . . .N-1)

$$\pi_o = \begin{cases} \left[1 + \sum_{k=1}^{N-1}\left\{\xi_k(r_k - r_{k+1}) - \frac{h_o\overline{g_k}}{b_k}\right\} + \frac{\xi_{N-1}}{1-e^{r}N}\left(1 + \frac{b_N}{h_N}\right]^{-1} & \text{if } b_k \neq 0 \\ \left[1 + \sum_{k=1}^{N-1}\left(\xi_k - \frac{h_o\overline{g_k}}{a_k}\right) + \frac{\xi_{N-1}}{2}\left(1 + \frac{a_N}{h_N}\right]^{-1} & \text{if } b_k = 0 \end{cases}$$

IV. SOME RESULTS

Let L be the mean number of customers in the system (including those in service). Then

$$L = \int_0^N x\, p(x)\, dx + N\pi_N$$

$$= \begin{cases} \sum_{k=1}^{N-1}[\xi_k(r_k - r_{k+1})(k - r_k - r_{k+1}) -] \frac{h_o\overline{g_k}}{b_k}\left(k - r_k - \frac{1}{2} - r_N\right) + \pi_N\left[\frac{b_N}{h_N}\left(N - \frac{1}{2} - r_N\right) + N\right] & \text{if } b_k \neq 0 \\ \pi_o \sum_{k=1}^{N-1}\left[\xi_k - \frac{h_o\overline{g_k}}{a_k}\left(k - \frac{1}{2}\right) - \frac{1}{6}\frac{h_o\overline{g_k}}{a_k}\right] + \pi_N\left[\frac{b_N}{h_N}\left(N - \frac{2}{3} + N\right]\right. & \text{if } b_k = 0 \end{cases}$$

The delay probability is given by

$DLY = 1 - P_{id}$

where,

$P_{id} \sum_{k=0}^{m-1} \pi_k$ is the probability that at least one server is not occupied

Thus

$$DLY = \begin{cases} 1 - \pi_o\left[1 + \sum_{k=1}^{N-1}\left\{r_k(\xi_k - \xi_k - 1) - \frac{h_o\overline{g_k}}{b_k}\right\}\right] & \text{if } b_k \neq 0 \\ 1 - \pi_o\left[1 + \sum_{k=1}^{N-1}\left\{\xi_k - \frac{h_o\overline{g_k}}{b_k}\right\}\right] & \text{if } b_k = 0 \end{cases}$$

the mean queue length (excluding customers in services) can be derived by using the formula

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ISSN: 2394-4404

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